LEIF MEJLBRO

REAL FUNCTIONS OF SEVERAL VARIABLES – NABLA...

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Real Functions of Several Variables

Examples of Nabla Calculus, Vector Potentials, Green's Identities and Curvilinear Coordinates, Electromagnetism and Various other Types

Calculus 2c-10

Real Functions of Several Variables – Examples of Nabla Calculus, Vector Potentials, Green's Identities and Curvilinear Coordinates, Electromagnetism and Various other Types – Calculus 2c-10

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Preface

In this volume I present some examples of *napla calculus, vector potentials, Green's identities, curvilinear coordinates , Electromagnetism and various other types*, cf. also *Calculus 2b, Functions of Several Variables*. Since my aim also has been to demonstrate some solution strategy I have as far as possible structured the examples according to the following form

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

One is used to from high school immediately to proceed to **I**. *Implementation*. However, examples and problems at university level are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be performed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. Instead, write in a plain language what you mean or want to do.

It is my hope that these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
15th October 2007

1 Nabla calculus

Example 1.1 *Let* $\mathbf{V}$ *denote a vector field, which is both divergence free and rotation free, and let* $\mathbf{e}$ *be a fixed unit vector. We consider also the following fields,*

$$F = -\mathbf{e}\cdot\mathbf{V}, \qquad \mathbf{W} = \mathbf{V}\times\mathbf{e}, \qquad \mathbf{U} = -\nabla F, \qquad \mathbf{T} = \nabla\times\mathbf{W}.$$

1) show that

$$\nabla\times(\mathbf{V}\times\mathbf{x}) = \mathbf{V} + \nabla(\mathbf{V}\cdot\mathbf{x}).$$

2) Show that $\mathbf{T}$ *is the same vector field as* $\mathbf{U}$*, and that this field also is both divergence free and rotation free.*

A Nabla calculus.

D Just exploit the assumptions,

$$\text{div }\mathbf{V} = \nabla\cdot\mathbf{V} = 0 \quad\text{and}\quad \mathbf{rot}\ \mathbf{V} = \nabla\times\mathbf{V} = \mathbf{0},$$

and the rules of differentiation of products.

I 1) We shall use the following well-known rule of calculation

$$\nabla \times (\mathbf{V} \times \mathbf{W}) = (\mathbf{W} \cdot \nabla)\mathbf{V} - \mathbf{W}(\nabla \cdot \mathbf{V}) - (\mathbf{V} \cdot \nabla)\mathbf{W} + \mathbf{V}(\nabla \cdot \mathbf{W})$$

with $\mathbf{W} = \mathbf{x}$, thus

$$\begin{aligned}\nabla \times (\mathbf{V} \times \mathbf{x}) &= (\mathbf{x} \cdot \nabla)\mathbf{V} - \mathbf{x}(\nabla \cdot \mathbf{V}) - (\mathbf{V} \cdot \nabla)\mathbf{x} + \mathbf{V}\,(\nabla \cdot \mathbf{x}) \\ &= (\mathbf{x} \cdot \nabla)\mathbf{V} - \mathbf{0} - (\mathbf{V} \cdot \nabla)\mathbf{x} + 3\mathbf{V} \\ &= \mathbf{V} + (\mathbf{x} \cdot \nabla)\mathbf{V} - (\mathbf{V} \cdot \nabla)\mathbf{x} + 2(\mathbf{V} \cdot \nabla)\mathbf{x} \\ &= \mathbf{V} + (\mathbf{x} \cdot \nabla)\mathbf{V} + (\mathbf{V} \cdot \nabla)\mathbf{x} \\ &= \mathbf{V} + \nabla(\mathbf{V} \cdot \mathbf{x}),\end{aligned}$$

where we have used that

$$(\mathbf{V} \cdot \nabla)\mathbf{x} = \left\{ V_1 \frac{\partial}{\partial x} + V_2 \frac{\partial}{\partial y} + V_3 \frac{\partial}{\partial z} \right\} (x, y, z) = (V_1, V_2, V_3) = \mathbf{V},$$

and that

$$\nabla(\mathbf{V} \cdot \mathbf{x}) = (\mathbf{V} \cdot \nabla)\mathbf{x} + (\mathbf{x} \cdot \nabla)\mathbf{V}.$$

2) Consider in particular $\mathbf{T}$ and put $\mathbf{W} = \mathbf{x}$. Then

$$\begin{aligned}\mathbf{T} &= \nabla \times \mathbf{W} = \nabla \times (\mathbf{V} \times \mathbf{e}) \\ &= (\mathbf{e} \cdot \nabla)\mathbf{V} - \mathbf{e}(\nabla \cdot \mathbf{V}) - (\mathbf{V} \cdot \nabla)\mathbf{e} + \mathbf{V}(\nabla \cdot \mathbf{e}) \\ &= (\mathbf{e} \cdot \nabla)\mathbf{V} - \mathbf{0} - \mathbf{0} + \mathbf{0} = (\mathbf{e} \cdot \nabla)\mathbf{V} \\ &= (\mathbf{e} \cdot \nabla)\mathbf{V} + (\mathbf{V} \cdot \nabla)\mathbf{e} = \nabla(\mathbf{e} \cdot \mathbf{V}) = -\nabla F = \mathbf{U},\end{aligned}$$

and the first claim is proved.
Since $\mathbf{T} = \mathbf{U} = -\nabla F$ is a gradient field, is is rotation free,

$$\nabla \times \mathbf{T} = -\nabla \times \nabla F = \mathbf{0}.$$

Since $\mathbf{T} = \mathbf{U} = \nabla \times \mathbf{W}$ is a rotation field, is is divergence free:

$$\nabla \cdot \mathbf{T} = \nabla \cdot \nabla \times \mathbf{W} = \mathbf{0}.$$

Example 1.2 *Let f be a C^1-function in r $(= \sqrt{x^2+y^2+z^2})$. We shall also (cf. the short hand notation in connection with the chain rule) consider f as a composed function $f(r(x,y,z))$, where $(x,y,z) \neq (0,0,0)$.*

1) Express ∇f by the derivative f' and $\mathbf{x}$.

2) Then set up formulæ for $\nabla \times (\mathbf{x}\, f)$ and for $\nabla \cdot (\mathbf{x}\, f)$.

3) Find the integer n, for which $\nabla \cdot (r^n\, \mathbf{x}) = 0$.

A Nabla calculus.

D Just follow the guidelines.

I We shall of course always assume that $r \neq 0$. Then

$$\nabla r = \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{1}{r}\,\mathbf{x}.$$

1) We get by the chain rule,

$$\nabla f = \left(f'(r)\,\frac{\partial r}{\partial x}, f'(r)\,\frac{\partial r}{\partial y}, f'(t)\,\frac{\partial r}{\partial z}\right) = f'(t)\;\nabla\, r = \frac{f'(r)}{r}\,\mathbf{x}.$$

2) A direct computation gives

$$\nabla \times (\mathbf{x}\, f) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x\, f(r) & y\, f(r) & z\, f(r) \end{vmatrix}$$

$$= \left(z\,\frac{f'(r)}{r}\,y - y\,\frac{f'(r)}{r}\,z, x\,\frac{f'(r)}{r}\,z - z\,\frac{f'(r)}{r}\,x, y\,\frac{f'(r)}{r}\,x - x\,\frac{f'(r)}{r}\,y\right) = (0,0,0).$$

A variant is

$$\begin{aligned} \nabla \times (\mathbf{x}\, f) &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x\, f(r) & y\, f(r) & z\, f(r) \end{vmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial f(r)}{\partial x} & \dfrac{\partial f(r)}{\partial y} & \dfrac{\partial f(r)}{\partial z} \\ x & y & z \end{vmatrix} \\ &= \nabla f \times \mathbf{x} = \frac{f'(r)}{r}\,\mathbf{x} \times \mathbf{x} = \mathbf{0}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \nabla \cdot (\mathbf{x}\, f) &= \left(f(r) + x\,\frac{\partial f}{\partial x}\right) + \left(f(r) + y\,\frac{\partial f}{\partial y}\right) + \left(f(r) + z\,\frac{\partial f}{\partial z}\right) \\ &= 3\, f(r) + \mathbf{x}\cdot\nabla f = 3f(r) + \frac{f'(r)}{r}\,\mathbf{x}\cdot\mathbf{x} \\ &= 3f(r) + r\, f'(r). \end{aligned}$$

3) Choose $f(r) = r^n$. Then it follows from the above,

$$\nabla \cdot (r^n\,\mathbf{x}) = 3r^n + n\, r^{n-1} = (3+n)r^n.$$

When $r \neq 0$, this is equal to 0 for $n = -3$. REMARK. In general, $\nabla \cdot (\mathbf{x}\, f(r)) = 0$ generates the differential equation

$$r\, f'(r) + 3f(r) = 0.$$

Then by separation of the variables,

$$\frac{f'(r)}{f(r)} \left[= \frac{\ln|f(r)|}{dr}\right] = -\frac{3}{r},$$

and the complete solution is obtained by an integration,

$$f(r) = C\cdot r^{-3}, \qquad r \neq 0, \quad \text{where } C \in \mathbb{R}. \qquad \Diamond$$

Example 1.3 *Let* $\mathbf{a}$ *be a constant vector, and let* f *be a* C^1*-function in one variable. We define*

$$g(\mathbf{x}) = f(\mathbf{a} \cdot \mathbf{x}).$$

1) Express the gradient $\bigtriangledown g$ *by the derivative* f'.
(Use one of the special cases of the chain rule).

2) Let also $\mathbf{V}$ *be a gradient field, and let* $k = 3$*. Show that the vector* $\bigtriangledown \times (g\mathbf{V})$ *is perpendicular to both* $\mathbf{a}$ *and* $\mathbf{V}$.

A Nabla calculus.

D Just compute.

I 1) If $\mathbf{a} = (a_1, \dots, a_k)$ and $\mathbf{x} = (x_1, \dots, x_k)$, then

$$g(\mathbf{x}) = f(\mathbf{a} \cdot \mathbf{x}) = f\left(\sum_{j=1}^{k} a_j\, x_j\right),$$

thus

$$\frac{\partial g}{\partial x_j} = f'(\mathbf{a} \cdot \mathbf{x})\, a_j,$$

hence

$$\bigtriangledown g = f'(\mathbf{a} \cdot \mathbf{x})\, \mathbf{a}.$$

2) If $\mathbf{V}$ is a gradient field, then there exists a function F, such that $\mathbf{V} = \bigtriangledown F$. Hence,

$$\begin{aligned} \bigtriangledown \times (g\,\mathbf{V}) &= (\bigtriangledown g) \times \mathbf{V} + g\, \bigtriangledown \times \mathbf{V} \\ &= f'(\mathbf{a} \cdot \mathbf{x})\, \mathbf{a} \times \mathbf{V} + f(\mathbf{a} \cdot \mathbf{x})\, \bigtriangledown \times (\bigtriangledown F) \\ &= f'(\mathbf{a} \cdot \mathbf{x})\, \mathbf{a} \times \mathbf{V} + \mathbf{0} \\ &= f'(\mathbf{a} \cdot \mathbf{V}, \end{aligned}$$

which shows that $\bigtriangledown \times (g\,\mathbf{V})$ is perpendicular on both $\mathbf{a}$ and $\mathbf{V}$.

Example 1.4 *Show the formula*

$$2(\bigtriangledown f) \cdot (\bigtriangledown \times (f\,\mathbf{V})) = (\bigtriangledown \times \mathbf{V}) \cdot \bigtriangledown (f^2).$$

A Nabla calculus.

D Just compute.

I We get straight away,

$$
\begin{aligned}
&2(\nabla f)\cdot(\nabla\times(f\,\mathbf{V}))\\
&\quad=2\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)\cdot\left(\frac{\partial}{\partial y}(fV_z)-\frac{\partial}{\partial z}(fV_y),\frac{\partial}{\partial z}(fV_x)-\frac{\partial}{\partial x}(fV_z),\frac{\partial}{\partial x}(fV_y)-\frac{\partial}{\partial y}(fV_x)\right)\\
&\quad=2\,\frac{\partial f}{\partial x}\left\{\frac{\partial f}{\partial y}\,V_z+f\,\frac{\partial V_z}{\partial y}-\frac{\partial f}{\partial z}\,V_y-f\,\frac{\partial V_y}{\partial z}\right\}\\
&\qquad\quad+2\,\frac{\partial f}{\partial y}\left\{\frac{\partial f}{\partial z}\,V_x+f\,\frac{\partial V_x}{\partial z}-\frac{\partial f}{\partial x}\,V_z-f\,\frac{\partial V_z}{\partial x}\right\}\\
&\qquad\quad+2\,\frac{\partial f}{\partial z}\left\{\frac{\partial f}{\partial x}\,V_y+f\,\frac{\partial V_y}{\partial x}-\frac{\partial f}{\partial y}\,V_x-f\,\frac{\partial V_x}{\partial y}\right\}\\
&\quad=\frac{\partial(f^2)}{\partial x}\left\{\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right\}+\frac{\partial(f^2)}{\partial y}\left\{\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right\}+\frac{\partial(f^2)}{\partial z}\left\{\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right\}\\
&\quad=\nabla(f^2)\cdot(\nabla\times\mathbf{V}).
\end{aligned}
$$

Example 1.5 *Let* $\mathbf{V}$ *be a* C^1 *vector field in the set* $A \subseteq \mathbb{R}^3$*. Show that if there exists a* C^1 *function* $g : A \to \mathbb{R} \setminus \{0\}$*, such that* $g\,\mathbf{V}$ *is a gradient field in* A*, then*

$$\mathbf{V} \cdot (\nabla \times \mathbf{V}) = 0$$

in the set A*.*

A Nabla calculus.

D Start by analyzing the assumption. Compute $\nabla \times \mathbf{V}$ by means of the rules of calculations.

I The assumption assures that there exists a C^2 function F, such that

$$g\,\mathbf{V} = \nabla F, \quad \text{i.e.} \quad \mathbf{V} = \frac{1}{g} \nabla F = h \nabla F,$$

where $h : A \to \mathbb{R} \setminus \{0\}$ is C^1, because $g(\mathbf{x}) \neq 0$. Then

$$\begin{aligned} \nabla \times \mathbf{V} &= \nabla \times (h \nabla F) \\ &= (\nabla h) \times \nabla F + h \nabla \times \nabla F \\ &= (\nabla h) \times \nabla F, \qquad \text{[the rotation of a gradient is } \mathbf{0}]. \end{aligned}$$

Now, ∇F is perpendicular to $(\nabla h) \times (\nabla F)$, hence

$$\mathbf{V} \cdot (\nabla \times \mathbf{V}) = h \nabla F \cdot \{(\nabla h) \times \nabla F\} = 0.$$

Example 1.6 *Let* α *be a constant. Find* $\nabla(r^\alpha)$ *and* $\nabla^2(r^\alpha)$*.*

A Nabla calculus.

D Just compute.

I When $r \neq 0$, then

$$\nabla r = \frac{1}{r}(x, y, z),$$

hence by the chain rule,

$$\nabla(r^\alpha) = \alpha\, r^{\alpha-1} \nabla r = \alpha\, r^{\alpha-2}(x, y, z) = \alpha\, r^{\alpha-2}\,\mathbf{x}.$$

By taking the divergence we get

$$\begin{aligned} \nabla^2(r^\alpha) &= \nabla \cdot \nabla(r^\alpha) = \nabla \cdot \left\{\alpha\, r^{\alpha-2}(x, y, z)\right\} \\ &= \alpha(\alpha-2)\, r^{\alpha-4}(x, y, z) \cdot (x, y, z) + 3\alpha\, r^{\alpha-2} \\ &= \alpha(\alpha-2)\, r^{\alpha-4} \cdot r^2 + 3\alpha\, r^{\alpha-2} \\ &= \alpha(\alpha+1)\, r^{\alpha-2}. \end{aligned}$$

Example 1.7 *Let* $\mathbf{e}$ *be a constant unit vector. Show that*

$$\mathbf{e}\cdot\{\nabla(\mathbf{V}\cdot\mathbf{e})-\nabla\times(\mathbf{V}\times\mathbf{e})\}=\nabla\cdot\mathbf{V}.$$

A Nabla calculus.

D Just compute.

I We get by means of the rules of calculation,

$$\begin{aligned}&\mathbf{e}\cdot\{\nabla(\mathbf{V}\cdot\mathbf{e})-\nabla\times(\mathbf{V}\times\mathbf{e})\}\\
&\quad=\mathbf{e}\cdot\{[(\mathbf{e}\cdot\nabla)\mathbf{V}+\mathbf{e}\times(\nabla\times\mathbf{V})+(\mathbf{V}\cdot\nabla)\mathbf{e}+\mathbf{V}\times(\nabla\times\mathbf{e})]-\nabla\times(\mathbf{V}\times\mathbf{e})\}\\
&\quad=\mathbf{e}\cdot\{(\mathbf{e}\cdot\nabla)\mathbf{V}+[\mathbf{e}\times(\nabla\times\mathbf{V})]-\nabla\times(\mathbf{V}\times\mathbf{e})\}\\
&\quad=\mathbf{e}\cdot\{(\mathbf{e}\cdot\nabla)\mathbf{V}-\nabla\times(\mathbf{V}\times\mathbf{e})\}+\mathbf{e}\cdot[\mathbf{e}\times(\nabla\times\mathbf{V})]\\
&\quad=\mathbf{e}\cdot\{(\mathbf{e}\cdot\nabla)\mathbf{V}-[(\mathbf{e}\cdot\nabla)\mathbf{V}-\mathbf{e}(\nabla\cdot\mathbf{V})-(\mathbf{V}\cdot\nabla)\mathbf{e}+\mathbf{V}(\nabla\cdot\mathbf{e})]\}+0\\
&\quad=\mathbf{e}\cdot\mathbf{e}(\nabla\cdot\mathbf{V})+0\\
&\quad=\nabla\cdot\mathbf{V}.\end{aligned}$$

This formula can of course also be written in the form

$$\mathbf{e}\cdot\{\mathbf{grad}(\mathrm{div}\ \mathbf{e})-\mathbf{rot}(\mathbf{V}\times\mathbf{e})\}=\mathrm{div}\ \mathbf{V}.$$

Example 1.8 *Let* α *be a constant vector. For* $\mathbf{x}\neq\mathbf{0}$ *we consider the fields*

$$U(\mathbf{x})=\frac{\mathbf{a}\cdot\mathbf{x}}{\|\mathbf{x}\|^3},\qquad \mathbf{W}(\mathbf{x})=\frac{\mathbf{a}\times\mathbf{x}}{\|\mathbf{x}\|^3}.$$

Show that

$$\nabla\times\mathbf{W}=-\nabla U.$$

A Nabla calculus.

D Just compute by using the rulse of calculation and the result of **Example 1.6**.

I Clearly, U and $\mathbf{W}$ are C^∞ for $\mathbf{x}\neq\mathbf{0}$. Put $r=\|\mathbf{x}\|$. Then by **Example 1.6**,

$$\nabla(r^\alpha)=\alpha\, r^{\alpha-2}\,\mathbf{x}\qquad\text{for }\mathbf{x}\neq\mathbf{0}.$$

Then we shall use the following result from Linear Algebra,

$$\mathbf{x}\times(\mathbf{a}\times\mathbf{x})=(\mathbf{x}\cdot\mathbf{x})\mathbf{a}-(\mathbf{a}\cdot\mathbf{x})\mathbf{x}=r^2\,\mathbf{a}-(\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}.$$

Applying these preparations we get

$$\begin{aligned}
\nabla \times \mathbf{W} &= \nabla \times \left\{ r^{-3}(\mathbf{a}\times\mathbf{x})\right\} && \text{definition of } \mathbf{W}\\
&= (\nabla r^{-3}) \times (\mathbf{a}\times\mathbf{x} + r^{-3}\ \nabla \times (\mathbf{a}\times\mathbf{x}) && \text{rule of calculation}\\
&= -3r^{-5}\mathbf{x}\times(\mathbf{a}\times\mathbf{x}) + r^{-3}\ \nabla\times(\mathbf{a}\times\mathbf{x}) && \textbf{Example 1.6}\\
&= -3r^{-5}\{r^2\mathbf{a} - (\mathbf{a}\cdot\mathbf{x})\mathbf{x}\} + r^{-3}\ \nabla\times(\mathbf{a}\times\mathbf{x}) && \text{Linear Algebra}\\
&= -\frac{3}{r^3}\,\mathbf{a} + \frac{3\mathbf{a}\cdot\mathbf{x}}{r^5}\,\mathbf{x} + \frac{1}{r^3}\{0-0-(\mathbf{a}\cdot\nabla)\mathbf{x}+\mathbf{a}(\nabla\cdot\mathbf{x})\} && \text{rule of computation}\\
&= -\frac{3}{r^3}\,\mathbf{a} + \frac{3}{r^5}\,(\mathbf{a}\cdot\mathbf{x})\mathbf{x} + \frac{1}{r^3}\,(-\mathbf{a}+3\mathbf{a}) && \text{computation}\\
&= -\frac{1}{r^3}\,\mathbf{a} + \frac{3}{r^5}\,(\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}, && \text{reduction,}
\end{aligned}$$

and

$$\begin{aligned}
\nabla U &= \nabla\left(r^{-3}(\mathbf{a}\cdot\mathbf{x})\right) && \text{definition of } U\\
&= (\mathbf{a}\cdot\mathbf{x})\ \nabla\left(r^{-3}\right) + \frac{1}{r^3}\ \nabla\,(\mathbf{a}\cdot\mathbf{x}) && \text{rule of calculation}\\
&= (\mathbf{a}\cdot\mathbf{x})\cdot\left\{-\frac{3}{r^5}\,\mathbf{x}\right\} + \frac{1}{r^3}\,\mathbf{a} && \textbf{Example 1.6} \text{ and } \nabla\,(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a}.
\end{aligned}$$

It follows by a comparison of these two expressions that

$$\nabla \times \mathbf{W} = -\nabla\, U.$$

This can also be written

$$\mathbf{rot}\ \mathbf{W} = -\mathbf{grad}\ U,$$

where U and $\mathbf{W}$ are given above.

Example 1.9 *Consider the composite vector function*

$$\mathbf{V}(\mathbf{x}) = \mathbf{U}(w), \qquad w = f(\mathbf{x}).$$

Find an expression for $\nabla\cdot\mathbf{V}$ *and* $\nabla\times\mathbf{V}$.

A Nabla calculus.

D Just compute.

I In general,

$$\frac{\partial V_j}{\partial x_i} = \frac{\partial(U_j\circ f)}{\partial x_i} = \frac{dU_j}{dw}\cdot\frac{\partial f}{\partial x_i}, \qquad w = f(\mathbf{x}).$$

When we change notation $(x_1, x_2, x_3) = (x, y, z)$, it follows that

$$\nabla \cdot \mathbf{V} = \sum_{i=1}^{3} U_i'(w)\,\frac{\partial f}{\partial x_i} = \nabla f \cdot \mathbf{U}'(f(\mathbf{x})),$$

and

$$\begin{aligned}
\nabla \times \mathbf{V} &= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}, \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \\
&= \begin{pmatrix} U_z'(w)\,\dfrac{\partial f}{\partial y} - U_y'(w)\,\dfrac{\partial f}{\partial z} \\[2ex] U_x'(w)\,\dfrac{\partial f}{\partial z} - U_z'(w)\,\dfrac{\partial f}{\partial x} \\[2ex] U_y'(w)\,\dfrac{\partial f}{\partial x} - U_x'(w)\,\dfrac{\partial f}{\partial y} \end{pmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\[1ex] \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\[2ex] U_x' & U_y' & U_z' \end{vmatrix} = \nabla f \times \mathbf{U}'(f(\mathbf{x})).
\end{aligned}$$

Alternatively, a more sophisticated reasoning is the following,

$$\bigtriangledown \times \mathbf{V} = \left| \begin{array}{ccc} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ U_x \circ f & U_y \circ f & U_z \circ f \end{array} \right| = \left| \begin{array}{ccc} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ U_x' \circ f & U_y' \circ f & U'z \circ f \end{array} \right| = \bigtriangledown f \times \mathbf{U}'(f(\mathbf{x})).$$

Summarizing we obtain the results

$$\bigtriangledown \cdot (\mathbf{U} \circ f(\mathbf{x})) = \bigtriangledown f(\mathbf{x}) \cdot \mathbf{U} - (f(\mathbf{x})) \qquad \text{and} \qquad \bigtriangledown \times (\mathbf{U} \circ f(\mathbf{x})) = \bigtriangledown f(\mathbf{x}) \times \mathbf{U}'(f(\mathbf{x})).$$

Example 1.10 *Given a C^1 vector field $\mathbf{V}$ and a C^2 scalar field f with the following property: The vector $\mathbf{V}$ is at each point (x, y, z) perpendicular to the level surface of f through the point (x, y, z). Prove that $\mathbf{V} \cdot (\bigtriangledown \times \mathbf{V}) = 0$.*

A Nabla calculus.

D Analyze the assumption. Then find a relation between f and $\mathbf{V}$. Finally, compute $\mathbf{V} \cdot (\bigtriangledown \times \mathbf{V})$.

I Since both $\bigtriangledown f$ and $\mathbf{V}$ are perpendicular to the level surface, they are proportional at each point. Hence, there exists a function g, such that (usually)

(1) $\mathbf{V}(x, y, z) = g(x, y, z) \bigtriangledown f(x, y, z)$.

When $\bigtriangledown f \neq \mathbf{0}$, then clearly g is os class C^1. Thus, when $\bigtriangledown f \neq \mathbf{0}$, then

$$\bigtriangledown \times \mathbf{V} = \bigtriangledown \times (g \bigtriangledown f) = (\bigtriangledown g) \times (\bigtriangledown f) + g\,(\bigtriangledown \times \bigtriangledown f) = (\bigtriangledown g) \times (\bigtriangledown f) + \mathbf{0}.$$

Since $\bigtriangledown f$ is perpendicular to $\bigtriangledown g \times \bigtriangledown f$, we get

$$\mathbf{V} \cdot (\bigtriangledown \times \mathbf{V}) = g \bigtriangledown f \cdot \{\bigtriangledown g \times \bigtriangledown f\} = 0.$$

If $\bigtriangledown f = 0$, then (1) does not necessary hold. However, if (1) holds, the relation is trivial.

Now assume that (1) does not hold, i.e. $\mathbf{V}(x, y, z) \neq \mathbf{0}$ and $\bigtriangledown f(x, y, z) = \mathbf{0}$. We shall then use a continuity argument:

Since f has level surfaces, we must have $\bigtriangledown f \neq \mathbf{0}$ arbitrarily close to (x, y, z), and it follows from the above that $\mathbf{V} \cdot (\bigtriangledown \times \mathbf{V}) = 0$ at these points. This relation is continuous, so it follows by a continuous extension that $\mathbf{V} \cdot \{\bigtriangledown \times \mathbf{V}\} = 0$ also is valid at points, where $\bigtriangledown f(x, y, z) = \mathbf{0}$.

Example 1.11 *Show by means of* Gauß's theorem *that for any* closed *surface $\mathcal{F}$,*

$$\int_{\mathcal{F}} \mathbf{n}\, dS = \mathbf{0}.$$

A Gauß's theorem in its operator version.

D Insert the obvious into Gauß's theorem in its operator version.

I Let $\mathcal{F}$ be the boundary of the domain Ω. Then by Gauß's theorem in its operator version,

$$\int_{\Omega} \nabla \Box \, d\Omega = \int_{\partial\Omega} \mathbf{n} \Box \, dS = \int_{\mathbf{F}} \mathbf{n} \Box \, dS.$$

If we replace $\Box$ we 1, it follows that

$$\int_{\mathcal{F}} \mathbf{n} \, dS = \int_{\Omega} \nabla 1 \, d\Omega = \int_{\Omega} \mathbf{0} \, d\Omega = \mathbf{0}.$$

Example 1.12 *Find the divergence of the vector field*

$$\mathbf{V} = (\nabla f) \times (\nabla g).$$

[Cf. **Example 2.3***.]*

A Nabla calculus.

D Just compute.

I The rotation of a gradient is $\mathbf{0}$, i.e. every gradient field is rotation free. Hence

$$\nabla \cdot (\nabla f \times \nabla g) = (\nabla \times \nabla f) \cdot \nabla g - (\nabla \times \nabla g) \cdot \nabla f = 0 - 0 = 0.$$

Example 1.13 *Consider the vector field* $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ *given by*

$$\mathbf{V}(x, y, z) = f(x, y) \, \mathbf{e}_z,$$

which also satisfies

$$\nabla \times (\nabla \times \mathbf{V}) = \alpha \, \mathbf{V},$$

where α *is a constant. Find a differential equation which has the function* f *as one of its solutions.*

A Double rotation.

D Compute the left hand side.

I It follows from $\mathbf{V}(x, y, z) = f(x, y) \, \mathbf{e}_z$ that

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & f(x, y) \end{vmatrix} = \left(f_y', -f_x', 0 \right)$$

and

$$\nabla \times (\nabla \times \mathbf{V}) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_y' & -f_x' & 0 \end{vmatrix} = \left(0 \, , \, 0 \, , -f_{xx}'' - f_{yy}'' \right) = (0, 0, \nabla^2 f),$$

and

$$\alpha\,\mathbf{V} = (0,0,\alpha\,f(x,y)).$$

Then from $\bigtriangledown \times (\bigtriangledown \times \mathbf{V}) = \alpha\,\mathbf{V}$,

$$-\bigtriangledown^2 f = \alpha\,f \quad \text{or} \quad \bigtriangledown^2 f + \alpha\,f = \Delta f + \alpha\,f = 0.$$

Example 1.14 *Let V denote the volume of a domain Ω in space with the outwards unit normal vector field $\mathbf{n}$, and let $\mathbf{a}$ be a constant vector. Find*

$$\frac{1}{V}\int_{\partial\Omega} \mathbf{n} \times (\mathbf{x} \times \mathbf{a})\,dS.$$

A Nabla calculus.

D Apply a variant Gauß's theorem and use the nabla calculations.

I By a variant of Gauß's theorem,

$$\int_{\partial\Omega} \mathbf{n} \times \mathbf{V}\,dS = \int_{\Omega} \bigtriangledown \times \mathbf{V}\,d\Omega.$$

Put $\mathbf{V} = \mathbf{x} \times \mathbf{a}$. Then

$$\frac{1}{V}\int_{\partial\Omega} \mathbf{n} \times (\mathbf{x} \times \mathbf{a})\,dS = \frac{1}{V}\int_{\Omega} \bigtriangledown \times (\mathbf{x} \times \mathbf{a})\,dS.$$

Then by a rule of calculation,

$$\begin{aligned}
\bigtriangledown \times (\mathbf{x} \times \bigtriangledown a) &= (\mathbf{a}\cdot\bigtriangledown)\mathbf{x} - \mathbf{a}(\bigtriangledown\cdot\mathbf{x}) - (\mathbf{x}\cdot\bigtriangledown)\mathbf{a} + \mathbf{x}(\bigtriangledown\cdot\mathbf{a}) \\
&= (\mathbf{a}\cdot\bigtriangledown)\mathbf{x} - \mathbf{a}(\bigtriangledown\cdot\mathbf{x}) - \mathbf{0} + \mathbf{0} \\
&= \left(a_1\,\frac{\partial}{\partial x} + a_2\,\frac{\partial}{\partial y} + a_3\,\frac{\partial}{\partial z}\right)(x,y,z) - \mathbf{a}\cdot(1+1+1) \\
&= \mathbf{a} - 3\mathbf{a} = -2\mathbf{a},
\end{aligned}$$

which is a constant. Thus by insertion,

$$\frac{1}{V}\int_{\partial\Omega} \mathbf{n} \times (\mathbf{x} \times \mathbf{a})\,dS = \frac{1}{V}\int_{V} (-2\mathbf{a}\,dS = -2\mathbf{a}.$$

ADDITION. For completeness we here prove the variant of Gauß's theorem, which is applied above. First note that the usual version of Gauß's theorem can be written

$$\int_{\partial\Omega} \mathbf{n}\cdot\mathbf{W}\,dS = \int_{\Omega} \bigtriangledown\cdot\mathbf{W}\,d\Omega.$$

Choose $\mathbf{W} = \mathbf{V} \times \mathbf{b}$, where $\mathbf{b}$ is any constant vector. Then

$$(2) \quad \int_{\partial\Omega} \mathbf{n} \cdot (\mathbf{V} \times \mathbf{b})\, dS = \int_{\Omega} \nabla \cdot (\mathbf{V} \times \mathbf{b})\, d\Omega.$$

The geometric interpretation of $\mathbf{n} \cdot (\mathbf{V} \times \mathbf{b})$ is that it is equal to the (signed) volume of the parallelepiped defined by the vectors $\mathbf{n}$, $\mathbf{V}$ and $\mathbf{b}$. (This simple result is also known from Linear Algebra).

The same interpretation is true for $(\mathbf{n} \times \mathbf{V}) \cdot \mathbf{b}$ (with the same sign, because the sequence of the vectors is not changed), thus

$$\mathbf{n} \cdot (\mathbf{V} \times \mathbf{b}) = (\mathbf{n} \times \mathbf{V}) \cdot \mathbf{b}.$$

Since $\mathbf{b}$ is constant, it follows by a rule of calculation,

$$\nabla \cdot (\mathbf{V} \times \mathbf{b}) = (\nabla \times \mathbf{V}) \cdot \mathbf{b} - (\nabla \times \mathbf{b} \cdot \mathbf{V}) = (\nabla \times \mathbf{V}) \cdot \mathbf{b}.$$

By inserting these two results into (2), we get

$$\int_{\partial\Omega} (\mathbf{n} \times \mathbf{V}) \cdot \mathbf{b}\, dS = \int_{\Omega} (\nabla \times \mathbf{V}) \cdot \mathbf{b}\, d\Omega.$$

Since $\mathbf{b}$ is a constant vector, it follows by a rearrangement,

$$\left\{\int_{\partial\Omega} \mathbf{n}\times\mathbf{V}\,dS - \int_{\Omega} \bigtriangledown\times\mathbf{V}\,d\Omega\right\}\cdot\mathbf{b} = 0.$$

Now, $\mathbf{0}$ is the only vector, which is perpendicular to all vectors, thus the first factor must be $\mathbf{0}$, and we get by another rearrangement.

$$\int_{\partial\Omega} \mathbf{n}\times\mathbf{V}\,dS = \int_{\Omega} \bigtriangledown\times\mathbf{V}\,d\Omega,$$

and the variant of Gauß's theorem has been proved.

Example 1.15 *Let* $\mathbf{V}$, $\mathbf{W}$ *be vector fields in the space which also depend on time* t *and satisfy the equations*

$$\bigtriangledown\times\mathbf{V} = \alpha\,\frac{\partial\mathbf{W}}{\partial t}, \qquad \bigtriangledown\times\mathbf{W} = -\beta\,\frac{\partial\mathbf{V}}{\partial t},$$

where α *and* β *are constants. Show that the vector field*

$$\mathbf{U} = \beta\mathbf{V}\times\frac{\partial\mathbf{V}}{\partial t} + \alpha\mathbf{W}\times\frac{\partial\mathbf{W}}{\partial t}$$

and the scalar field

$$f = \beta\mathbf{V}\cdot\bigtriangledown\times\mathbf{V} + \alpha\mathbf{W}\cdot\bigtriangledown\times\mathbf{W}$$

satisfy the differential equation

$$\bigtriangledown\cdot\mathbf{U} + \frac{\partial f}{\partial t} = 0.$$

(An equation of this type is often called a continuity equation or a preservation theorem).

A Continuity equation.

D Nabla calculus.

I Since $\dfrac{\partial}{\partial t}$ is a differentiation with respect to a "parameter", where can interchange $\dfrac{\partial}{\partial t}$ with any of the operators $\bigtriangledown$, $\bigtriangledown\cdot$ and $\bigtriangledown\times$. Thus

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \beta\,\frac{\partial\mathbf{V}}{\partial t}\cdot(\bigtriangledown\times\mathbf{V}) + \beta\mathbf{V}\cdot\left(\bigtriangledown\times\frac{\partial\mathbf{V}}{\partial t}\right) + \alpha\,\frac{\partial\mathbf{W}}{\partial t}\cdot(\bigtriangledown\times\mathbf{W}) + \alpha\mathbf{W}\cdot\left(\bigtriangledown\times\frac{\partial\mathbf{W}}{\partial t}\right)\\
&= -(\bigtriangledown\times\mathbf{W})\cdot(\bigtriangledown\times\mathbf{V}) + \beta\mathbf{V}\cdot\left(\bigtriangledown\times\frac{\partial\mathbf{V}}{\partial t}\right) + (\bigtriangledown\times\mathbf{V})\cdot(\bigtriangledown\times\mathbf{W}) + \alpha\mathbf{W}\cdot\left(\bigtriangledown\times\frac{\partial\mathbf{W}}{\partial t}\right)\\
&= \beta\mathbf{V}\cdot\left(\bigtriangledown\times\frac{\partial\mathbf{V}}{\partial t}\right) + \alpha\mathbf{W}\cdot\left(\bigtriangledown\times\frac{\partial\mathbf{W}}{\partial t}\right).
\end{aligned}$$

By using this rule of calculation we get similarly,

$$
\begin{aligned}
\nabla \cdot \mathbf{U} &= \beta \nabla \cdot \left(\mathbf{V} \times \frac{\partial \mathbf{V}}{\partial t} \right) + \alpha \nabla \cdot \left(\mathbf{W} \times \frac{\partial \mathbf{W}}{\partial t} \right) \\
&= \beta (\nabla \times \mathbf{V}) \cdot \frac{\partial \mathbf{V}}{\partial t} - \beta \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) \cdot \mathbf{V} + \alpha (\nabla \times \mathbf{W}) \cdot \frac{\partial \mathbf{W}}{\partial t} - \alpha \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right) \cdot \mathbf{W} \\
&= -(\nabla \times \mathbf{V}) \cdot (\nabla \times \mathbf{W}) - \beta \mathbf{V} \cdot \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) + (\nabla \times \mathbf{V}) \cdot (\nabla \times \mathbf{W}) - \alpha \mathbf{W} \cdot \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right) \\
&= -\beta \mathbf{V} \cdot \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) - \alpha \mathbf{W} \cdot \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right)
\end{aligned}
$$

Finally, by adding these expressions we get

$$\nabla \cdot \mathbf{U} + \frac{\partial f}{\partial t} = 0.$$

2 Vector potentials

Example 2.1 *Prove in each of the following cases that the given vector field $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ is divergence free. The find a vector potential $\mathbf{W} : \mathbb{R}^3 \to \mathbb{R}^3$, such that $\mathbf{V} = \bigtriangledown \times \mathbf{W}$. (We may not necessarily consider the points where $xyz = 0$).*

1) $\mathbf{V}(x, y, z) = \left(\cosh(z^2), \cosh(x^2), \cosh(y^2)\right)$.

2) $\mathbf{V}(x, y, z) = (x^2y + z, xy^2 + z, -4xyz)$.

3) $\mathbf{V}(x, y, z) = (xz, yz, -z^2)$.

4) $\mathbf{V}(x, y, z) = \left(\dfrac{1}{1+y^2}, \dfrac{1}{1+z^2}, \dfrac{1}{1+x^2}\right)$.

5) $\mathbf{V}(x, y, z) = \left(\dfrac{\sin z}{z}, \dfrac{\sin x}{x}, \dfrac{\sin y}{y}\right)$.

6) $\mathbf{V}(x, y, z) = (\exp x, y \exp x, -2z \exp x)$.

A Vector potential.

D Clearly, the domain $\mathbb{R}^3$ is star shaped. First prove that the field is divergence free. Then compute

$$\mathbf{U}(\mathbf{x}) = \int_0^1 t\,\mathbf{V}(t\,\mathbf{x})\,dt$$

and

$$\mathbf{W}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) \times \mathbf{x} = -\mathbf{x} \times \int_0^1 t\,\mathbf{V}(t\,\mathbf{x})\,dt = \int_0^1 \mathbf{V}(t\,\mathbf{x}) \times (t\,\mathbf{x})\,dt.$$

Finally, check the result, i.e. show that

$$\bigtriangledown \times \mathbf{W} = \mathbf{V}.$$

I 1. Since each V_i does not depend on x_i, we clearly have that $\bigtriangledown \cdot \mathbf{V} = 0$.

Because of the symmetry it suffices to compute

$$\int_0^1 t \cosh\left((tu)^2\right)\,dt = \int_0^1 t \cosh\left(t^2u^2\right)\,dt = \frac{1}{2}\int_0^1 \cosh\left(\tau\,u^2\right)\,d\tau = \frac{1}{2}\,\frac{\sinh(u^2)}{u^2},$$

where $\dfrac{\sinh\alpha}{\alpha}$ in general in the following is interpreted as 1, when $\alpha = 0$. Then continue either by a direct calculation or by a continuous extension, i.e. by going to the limit.

It follows from the above that

$$\mathbf{U}(\mathbf{x}) = \int_0^1 t\,\mathbf{V}(t\,\mathbf{x})\,dt = \frac{1}{2}\left(\frac{\sinh(z^2)}{z^2}, \frac{\sinh(x^2)}{x^2}, \frac{\sinh(y^12)}{y^2}\right),$$

hence

$$\begin{aligned}\mathbf{W} &= \mathbf{U}\times\mathbf{x} = \frac{1}{2}\left|\begin{array}{ccc}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\sinh(z^2)}{z^2} & \dfrac{\sinh(x^2)}{x^2} & \dfrac{\sinh(y^2)}{y^2} \\ x & y & z\end{array}\right| \\ &= \frac{1}{2}\left(z\cdot\frac{\sinh(x^2)}{x^2}-\frac{\sinh(y^2)}{y}\,,\,x\cdot\frac{\sinh(y^2)}{y^2}-\frac{\sinh(z^2)}{z}\,,y\cdot\frac{\sinh(z^2)}{z^2}-\frac{\sinh(x^2)}{x}\right).\end{aligned}$$

C TEST. We have

$$\begin{aligned}\nabla\times\mathbf{W} &= \frac{1}{2}\left|\begin{array}{ccc}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ W_1(x,y,z) & W_2(x,y,z) & W_3(x,y,z)\end{array}\right| \\ &= \frac{1}{2}\begin{pmatrix}\dfrac{\sinh(z^2)}{z^2}+2\,\sinh(z^2)-\dfrac{\sinh(z^2)}{z^2} \\ \dfrac{\sinh(x^2)}{x^2}+2\,\sinh(x^2)-\dfrac{\sinh(x^2)}{x^2} \\ \dfrac{\sinh(y^2)}{y^2}+2\,\sinh(y^2)-\dfrac{\sinh(y^2)}{y^2}\end{pmatrix} \\ &= \left(\sinh(z^2)\,,\,\sinh(x^2)\,,\,\sinh(y^2)\right)=\mathbf{V}.\end{aligned}$$

The result is correct.

I 2. First compute

$$\operatorname{div}\mathbf{V}=\nabla\cdot\mathbf{V}=2xy+2xy-4xy=0,$$

thus the field is divergence free. Then

$$\begin{aligned}\mathbf{U}(\mathbf{x}) &= \int_0^1 t\,\mathbf{V}(t\,\mathbf{x})\,dt=\left(\int_0^1 t\{t^3x^2y+tz^2\}dt,\int_0^1 t\{t^3xy^2+tz\}dt,-4xyz\int_0^1 t\cdot t^3dt\right) \\ &= \left(\frac{1}{5}\,x^2y+\frac{1}{3}\,z\,,\,\frac{1}{5}\,xy^2+\frac{1}{3}\,z\,,\,-\frac{4}{5}\,xyz\right),\end{aligned}$$

and hence

$$\begin{aligned}
\mathbf{W} &= \mathbf{U}\times\mathbf{x} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{1}{5}\,x^2y+\frac{1}{3}\,z & \frac{1}{5}\,xy^2+\frac{1}{3}\,z & -\frac{4}{5}\,xyz \\ x & y & z \end{vmatrix} \\
&= \left(\frac{1}{5}\,xy^2z+\frac{1}{3}\,z^2+\frac{4}{5}\,xy^2z\,,\,-\frac{4}{5}\,x^2yz-\frac{1}{5}\,x^2yz-\frac{1}{3}\,z^2,\right. \\
&\qquad \left.\frac{1}{5}\,x^2y^2+\frac{1}{3}\,yz-\frac{1}{5}\,x^2y^2+\frac{1}{3}\,yz-\frac{1}{5}\,x^2y^2-\frac{1}{3}\,xz\right) \\
&= \left(xy^2z+\frac{1}{3}\,z^2\,,\,-x^2yz-\frac{1}{3}\,z^2\,,\,\frac{1}{3}\,yz-\frac{1}{3}\,xz\right) \\
&= z\left(xy^2+\frac{1}{3}\,z\,,\,-x^2y-\frac{1}{3}\,z\,,\,\frac{1}{3}\,y-\frac{1}{3}\,x\right).
\end{aligned}$$

C TEST. Here

$$\begin{aligned}
\nabla\times\mathbf{W} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2z+\frac{1}{3}\,z^2 & -x^2yz-\frac{1}{3}\,z^2 & \frac{1}{3}\,yz-\frac{1}{3}\,xz \end{vmatrix} \\
&= \left(\frac{1}{3}\,z+x^2y+\frac{2}{3}\,z, xy^2+\frac{2}{3}\,z+\frac{1}{3}\,z, -2xyz-2xyz\right) \\
&= (x^2y+z, xy^2+z, -4xyz) = \mathbf{V}.
\end{aligned}$$

Our result has proved to be correct.

I 3. Since

$$\operatorname{div}\mathbf{V} = \nabla\cdot\mathbf{V} = z+z-2z = 0,$$

the field is divergence free.

Furthermore,

$$\begin{aligned}
\mathbf{U}(\mathbf{x}) &= \int_0^1 t\,\mathbf{V}(t\,\mathbf{x})\,dt = \left(\int_0^1 t\cdot t^2xz\,dt, \int_0^1 t\cdot t^2yz\,dt, -\int_0^1 t\cdot t^2z^2\,dt\right) \\
&= \frac{1}{4}\,(xz, yz, -z^2) = \frac{1}{4}\,\mathbf{V}(x,y,z),
\end{aligned}$$

thus

$$\begin{aligned}
\mathbf{W} &= \mathbf{U}\times\mathbf{x} = \frac{1}{4}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ xz & yz & -z^2 \\ x & y & z \end{vmatrix} = \frac{z}{4}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x & y & -z \\ x & y & z \end{vmatrix} = \frac{z}{4}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x & y & -z \\ 0 & 0 & 2z \end{vmatrix} \\
&= \frac{z^2}{2}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y \\ x & y \end{vmatrix} = \frac{1}{2}\,(yz^2, -xz^2, 0).
\end{aligned}$$

C TEST. We get

$$\nabla \times \mathbf{W} = \frac{1}{2}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz^2 & -xz^2 & 0 \end{vmatrix} = \frac{1}{2}\left(2xz, 2yz, -z^2-z^2\right) = \left(xz, yz, -z^2\right) = \mathbf{V}.$$

We have thus tested our result.

I 4. Clearly, since each V_i is independent of x_i, we must have $\nabla \cdot \mathbf{V} = 0$.

Because of the symmetry it suffices to compute

$$\int_0^1 t \cdot \frac{1}{1+(tu)^2}\,dt = \frac{1}{2}\int_0^1 \frac{1}{1+\tau\,u^2}\,d\tau = \frac{1}{2}\,\frac{\ln(1+u^2)}{u^2},$$

where the result by continuous extension is interpreted as $\dfrac{1}{2}$ for $u = 0$. Hence

$$\mathbf{U}(\mathbf{x}) = \int_0^1 t\,\mathbf{V}(t\,\mathbf{x})\,dt = \frac{1}{2}\left(\frac{\ln(1+y^2)}{y^2}, \frac{\ln(1+z^2)}{z^2}, \frac{\ln(1+x^2)}{x^2}\right),$$

and thus

$$\mathbf{W} \;=\; \mathbf{U}\times\mathbf{x} = \frac{1}{2}\left|\begin{array}{ccc} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \\ \dfrac{\ln(1+y^2)}{y^2} & \dfrac{\ln(1+z^2)}{z^2} & \dfrac{\ln(1+x^2}{)}x^2 \\ \\ x & y & z \end{array}\right|$$

$$= \frac{1}{2}\left(\frac{\ln(1+z^2)}{z} - y\,\frac{\ln(1+x^2)}{x^2}\,,\;\frac{\ln(1+x^2)}{x} - z\,\frac{\ln(1+y^2)}{y^2}\,,\;\frac{\ln(1+y^2)}{y} - x\,\frac{\ln(1+z^2)}{z^2}\right).$$

C TEST. Here

$$\nabla\times\mathbf{W} \;=\; \frac{1}{2}\left|\begin{array}{ccc} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \\ W_1(x,y,z) & W_2(x,y,z) & W_3(x,y,z) \end{array}\right|$$

$$= \frac{1}{2}\begin{pmatrix} \dfrac{2}{1+y^2} - \dfrac{\ln(1+y^2)}{y^2} + \dfrac{\ln(1+y^2)}{y^2} \\ \\ \dfrac{2}{1+z^2} - \dfrac{\ln(1+z^2)}{z^2} + \dfrac{\ln(1+z^2)}{z^2} \\ \\ \dfrac{2}{1+x^2} - \dfrac{\ln(1+x^2)}{x^2} + \dfrac{\ln(1+x^2)}{x^2} \end{pmatrix} = \left(\frac{1}{1+y^2}\;\frac{1}{1+z^2}\,,\;\frac{1}{1+x^2}\right) = \mathbf{V}(x,y,z).$$

We have tested our result.

I 5. Interpret $\dfrac{\sin u}{u}$ as 1, when $u=0$. Then V_i is independent of x_i (same index i in both places), and the field is clearly divergence free. Due to the symmetry it suffices to compute

$$\int_0^1 t\cdot\frac{\sin(tu)}{tu}\,dt = \frac{1}{u}\int_0^1 \sin(tu)\,du = \frac{1-\cos u}{u^2} \qquad \text{for } u \neq 0,$$

and

$$\int_0^1 t\,dt = \frac{1}{2} \qquad \text{for } u = 0,$$

where we interpret $\dfrac{1-\cos u}{u^2}$ as $\dfrac{1}{2}$, when $u=0$. This is in agreement with the continuous extension. Thus

$$\mathbf{U}(\mathbf{x}) = \int_0^1 t\,\mathbf{V}(t\,\mathbf{x})\,dt = \left(\frac{1-\cos z}{z^2}\,,\;\frac{1-\cos x}{x^2}\,,\;\frac{1-\cos y}{y^2}\right),$$

and hence

$$\begin{aligned}\mathbf{W} &= \mathbf{U}\times\mathbf{x} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{1-\cos z}{z^2} & \dfrac{1-\cos x}{x^2} & \dfrac{1-\cos y}{y^2} \\ x & y & z \end{vmatrix} \\ &= \left(z\cdot\frac{1-\cos x}{x^2} - \frac{1-\cos y}{y}\,,\, x\cdot\frac{1-\cos y}{y^2} - \frac{1-\cos z}{z}\,,\, y\cdot\frac{1-\cos z}{z^2} - \frac{1-\cos x}{x}\right).\end{aligned}$$

C TEST. It follows that

$$\begin{aligned}\nabla\times\mathbf{W} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ W_1(x,y,z) & W_2(x,y,z) & W_3(x,y,z) \end{vmatrix} \\ &= \begin{pmatrix} \dfrac{1-\cos z}{z^2} + \dfrac{\sin z}{z} - \dfrac{1-\cos z}{z^2} \\ \dfrac{1-\cos x}{x^2} + \dfrac{\sin x}{x} - \dfrac{1-\cos x}{x^2} \\ \dfrac{1-\cos y}{y^2} + \dfrac{\sin y}{y} - \dfrac{1-\cos y}{y^2} \end{pmatrix} = \left(\frac{\sin z}{z}\,,\,\frac{\sin x}{x}\,,\,\frac{\sin y}{y}\right) = \mathbf{V},\end{aligned}$$

and we have checked our result.

I 6. Since

$$\operatorname{div}\mathbf{V} = \nabla\cdot\mathbf{V} = \exp x + \exp x - 2\,\exp x = 0,$$

the field is divergence free.

Then

$$\begin{aligned}\mathbf{U}(x,y,z) &= \int_0^1 t\,(\exp(tx), ty\,\exp(tx), -2tz\,\exp(tx))\,dt \\ &= \left(\int_0^1 t\,\exp(tx)\,dt\,,\, y\int_0^1 t^2\exp(tx)\,dt\,,\, -2z\int_0^1 t^2\exp(tx)\,dt\right) \\ &= \left(\frac{1}{x^2}\int_0^x \tau\cdot\exp(\tau)\,d\tau\,,\,\frac{y}{x^3}\int_0^x \tau^2\exp(\tau)\,d\tau\,,\,-\frac{2z}{x^3}\int_0^x \tau^2\exp(\tau)\,d\tau\right).\end{aligned}$$

A small computation gives

$$\int_0^x \tau\,\exp(\tau)\,d\tau = (x-1)e^x + 1$$

and

$$\int_0^x \tau^2 \exp(\tau)\,d\tau = (x^2-2x+2)e^x - 2,$$

hence by insertion,

$$\mathbf{U}(x,y,z) = \left(\frac{(x-1)e^x+1}{x^2}\,,\; y\cdot\frac{(x^2-2x+2)e^x-2}{x^3}\,,\; -2z\cdot\frac{(x^2-2x+2)e^x-2}{x^3}\right).$$

Then

$$\begin{aligned}
\mathbf{W}(\mathbf{x}) &= \mathbf{U}(\mathbf{x})\times\mathbf{x} \\
&= \left|\begin{array}{ccc} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{(x-1)e^x+1}{x^2} & y\cdot\frac{(x^2-2x+2)e^x-2}{x^3} & -2z\cdot\frac{(x^2-2x+2)e^x-2}{x^3} \\ x & y & z \end{array}\right| \\
&= \begin{pmatrix} yz\cdot\dfrac{(x^2-2x+2)e^x-2}{x^3} + 2yz\cdot\dfrac{(x^2-2x+2)e^x-2}{x^3} \\ -2z\cdot\dfrac{(x^2-2x+2)e^x-2}{x^2} + z\cdot\dfrac{(x-1)e^x+1}{x^2} \\ y\cdot\dfrac{(x-1)e^x+1}{x^2} - y\cdot\dfrac{(x^2-2x+2)e^x-2}{x^2} \end{pmatrix} \\
&= \begin{pmatrix} 3yz\cdot\dfrac{(x^2-2x+2)e^x-2}{x^3} \\ -z\cdot\dfrac{(2x^2-5x+5)e^x-5}{x^2} \\ -y\cdot\dfrac{(x^2-3x+3)e^x-3}{x^2} \end{pmatrix}.
\end{aligned}$$

C "Test". Even if the original expression of $\mathbf{V}(x,y,z)$ looks very simple, an insertion into the solution formula will give very difficult expressions with e.g. x^2 and x^3 in the denominator. We shall therefore not in this case test the result, i.e. compute

$$\nabla\times\mathbf{W} = \left|\begin{array}{ccc} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ W_1(x,y,z) & W_2(x,y,z) & W_3(x,y,z) \end{array}\right|.$$

Example 2.2 *Consider a vector field* $\mathbf{V} : A \to \mathbb{R}^2$*, where* A *is an open star shaped subset of the* (X, Y)*-plane. Furthermore, assume that the field* $\mathbf{V}$ *is divergence free.*

1) Prove that the vector field $\mathbf{e}_z \times \mathbf{V}$ *is rotation free and that there exists a scalar field* $W : A \to \mathbb{R}$*, such that* $W\,\mathbf{ez}$ *is a vector potential of* $\mathbf{V}$*.*

2) Prove that a level curve of W *is a field line of* $\mathbf{V}$*.*

A Vector potential.

D Analyze the text step by step and prove the claims in succession.

I 1) According to the assumption, $\mathbf{V} : A \to \mathbb{R}^2$ is a function of the variable (x, y), which satisfies

$$\operatorname{div} \mathbf{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} = 0.$$

Define a vector field $\tilde{\mathbf{V}}$ by

$$\tilde{\mathbf{V}}(x, y, z) = (V_1(x, y), V_2(x, y), 0)\,, \qquad (x, y, z) \in A \times \mathbb{R} = \tilde{A}.$$

Then $\tilde{A}$ is star shaped and $\tilde{\mathbf{V}}$ is also divergence free.

We shall in the following only write $\mathbf{V}$ instead of the more precise $\tilde{\mathbf{V}}$.

By one of the formulæ of differentiation of a product,

$$\begin{aligned}\nabla \times (\mathbf{e}_z \times \mathbf{V}) &= (\mathbf{V}\cdot\nabla)\mathbf{e}_z - \mathbf{V}(\nabla\cdot\mathbf{e}_z) - (\mathbf{e}_z\cdot\nabla)\mathbf{V} + \mathbf{e}_z(\nabla\cdot\mathbf{V})\\ &= \mathbf{0}+\mathbf{0}-\frac{\partial}{\partial z}\mathbf{V} + \operatorname{div}\mathbf{V}\cdot\mathbf{e}_z = \mathbf{0},\end{aligned}$$

and the vector field $\mathbf{e}_z \times \mathbf{V}$ is rotation free.

Thus there exists a scalar field $\tilde{W} : A \times \mathbb{R} \to \mathbb{R}$, such that

$$\nabla\tilde{W} = \left(\frac{\partial\tilde{W}}{\partial x}, \frac{\partial\tilde{W}}{\partial y}, \frac{\partial\tilde{W}}{\partial z}\right) = \mathbf{e}_z \times \mathbf{V} = (-V_2, V_1, 0)\,.$$

Since $\mathbf{V}$ is independent of z, also $\tilde{W} = W$ must be independent of z, thus we can choose a scalar field $W : A \to \mathbb{R}^2$, such that

$$\nabla W = (-V_2, V_1, 0) = \mathbf{e}_z \times \mathbf{V}.$$

Further,

$$\begin{aligned}\nabla \times (E\mathbf{e}_z) &= (\nabla W)\times\mathbf{e}_z + W\,\nabla\times\mathbf{e}_z = (\mathbf{e}_z\times\mathbf{V})\times\mathbf{e}_z = \begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ -V_2 & V_1 & 0\\ 0 & 0 & 1\end{vmatrix}\\ &= \begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y\\ -V_2 & V_1\end{vmatrix} = (V_1, V_2, 0) = \mathbf{V}.\end{aligned}$$

(ALTERNATIVELY we may twice apply the geometric interpretation of the cross product). This shows that $W\,\mathbf{e}_z$ is a vector potential for $\mathbf{V}$, and we have proved all the claims.

2) A level curve of W is given by

$$W(x,y) = c,$$

where the tangent field $\mathbf{U}(x,y)$ of the level curve satisfies

$$\nabla W \cdot \mathbf{U} = (\mathbf{e}_z \times \mathbf{V})\cdot\mathbf{U} = 0.$$

Clearly, this equation has the solution $\mathbf{U} = \mathbf{V}$, thus the level curve is also a field line of $\mathbf{V}$.

Example 2.3 *Let α be a constant, and let two vector fields on $\mathbb{R}^3$ be given in the following way:*

$$\mathbf{U} = (\triangledown f) \times (\triangledown g), \qquad \mathbf{W} = \alpha\,(f \triangledown g - f \triangledown f).$$

Show that one can choose α such that $\mathbf{W}$ is a vector potential for $\mathbf{U}$.
[Cf. **Example 1.12**.]

A Vector potential.

D Compute $\triangledown \times \mathbf{W}$ and compare with $\mathbf{U} = \triangledown f \times \triangledown g$.

I By the rules of calculations,

$$\begin{aligned}
\triangledown \times \mathbf{W} &= \alpha \triangledown \times(f \triangledown g) - \alpha \triangledown \times(g \triangledown f) \\
&= \alpha \triangledown f \times \triangledown g + \alpha f\,(\triangledown \times \triangledown g) - \alpha \triangledown g \times \triangledown f - \alpha g\,(\triangledown \times \triangledown f) \\
&= \alpha \triangledown \times \triangledown g + \mathbf{0} + \alpha \triangledown f \times \triangledown g + \mathbf{0} \\
&= 2\alpha \triangledown f \times \triangledown g = 2\alpha\,\mathbf{U}.
\end{aligned}$$

We see that if $\alpha = \dfrac{1}{2}$, then $\mathbf{W}$ is a vector potential for $\mathbf{U}$.

Example 2.4 *Let $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ be a given vector field. Find in each of the following cases the following vector fields:*

$$\mathbf{S}(\mathbf{x}) = \int_0^1 \tau\,\mathbf{V}(\mathbf{x}\,\tau)\,d\tau, \quad \mathbf{U}(\mathbf{x}) = -\mathbf{x} \times \mathbf{S}(\mathbf{x}), \quad \mathbf{W}(\mathbf{x}) = \triangledown \times \mathbf{U}(\mathbf{x}).$$

1) $\mathbf{V}(x,y,z) = (x,y,z)$.

2) $\mathbf{V}(x,y,z) = \left(x^2, y^2, z^2\right)$.

3) $\mathbf{V}(x,y,z) = \left(4x^2, 0, 0\right)$.

4) $\mathbf{V}(x,y,z) = (0, \cos y, 0)$.

A The standard formula of computation of a vector potential applied on non-divergence free vector fields. The example shall illustrate what can go wrong when the assumptions are not fulfilled.

D First note that the given fields are not divergence free. Then just compute.

I 1) First note that $\operatorname{div}\mathbf{V} = 3 \neq 0$, thus the vector potential does not exist.
We shall nevertheless compute the candidate of the “vector potential” according to the standard procedure. First,

$$\mathbf{S}(\mathbf{x}) = \int_0^1 \tau\,\mathbf{V}(\mathbf{x}\,\tau)\,d\tau = \int_0^1 \tau\,(x\tau, y\tau, z\tau)\,d\tau = (x,y,z)\int_0^1 \tau^2\,d\tau = \frac{1}{3}\,(x,y,z).$$

Then

$$\mathbf{U}(\mathbf{x}) = -\mathbf{x} \times \mathbf{S}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \times \mathbf{x} = \frac{1}{3}\,\mathbf{x} \times \mathbf{x} = \mathbf{0},$$

and thus

$$\mathbf{W}(\mathbf{x}) = \bigtriangledown \times \mathbf{V}(\mathbf{x}) = \mathbf{0} \neq \mathbf{V}(\mathbf{x}).$$

2) Here

$$\operatorname{div} \mathbf{V} = 2(x+y+z) \neq 0,$$

so the field is not divergence free.
By a direct computation,

$$\begin{aligned} \mathbf{S}(\mathbf{x}) &= \int_0^1 \tau\,\mathbf{V}(\mathbf{x}\,\tau)\,d\tau = \int_0^1 \tau\left(x^2\tau^2, y^2\tau^2, z^2\tau^2\right)\,d\tau \\ &= \left(x^2, y^2, z^2\right)\int_0^1 \tau^3\,d\tau = \frac{1}{4}\,\left(x^2, y^2, z^2\right) = \frac{1}{4}\,\mathbf{V}. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{U}(\mathbf{x}) &= -\mathbf{x} \times \mathbf{S}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \times \mathbf{x} = \frac{1}{4}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix} \\ &= \frac{1}{4}\,\left(y^2z - z^2y, z^2x - x^2z, x^2y - y^2x\right), \end{aligned}$$

hence,

$$\begin{aligned} \mathbf{W}(\mathbf{x}) &= \bigtriangledown \times \mathbf{U}(\mathbf{x}) = \frac{1}{4}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z - z^2y & z^2x - x^2z & x^2y - y^2x \end{vmatrix} \\ &= \frac{1}{4}\begin{pmatrix} x^2 - 2yz - 2xz + x^2 \\ y^2 - 2yz - 2yz + y^2 \\ z^2 - 2xz - 2yz + z^2 \end{pmatrix} = \frac{1}{2}\,\left(x^2 - x(y+z), y^2 - y(x+z), z^2 - z(x+y)\right), \end{aligned}$$

which clearly is different from $\mathbf{V}(\mathbf{x})$.

3) Here $\operatorname{div} \mathbf{V} = 8x \neq 0$, and the field is not divergence free.
By a direct computation,

$$\mathbf{S}(\mathbf{x}) = \int_0^1 \tau\,\mathbf{V}(\mathbf{x}\,\tau)\,d\tau = \int_0^1 \tau(4x^2\tau^2, 0, 0)\,d\tau = (x^2, 0, 0).$$

Then

$$\mathbf{U}(\mathbf{x}) = -\mathbf{x} \times \mathbf{S}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \times \mathbf{x} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x^2 & 0 & 0 \\ x & y & z \end{vmatrix} = \left(0, -x^2 z, x^2 y\right),$$

hence

$$\mathbf{W}(\mathbf{x}) = \nabla \times \mathbf{U}(\mathbf{x}) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & -x^2 z & x^2 y \end{vmatrix} = \left(x^2 + x^2, -2xy, -2xz\right) = 2x\,(x, -y, -z),$$

which clearly is not equal to $\mathbf{V}(\mathbf{x})$.

4) It follows immediately that div $\mathbf{V} = -\sin y \neq 0$, so the field is not divergence free.
Then by direct computation,

$$\mathbf{S}(\mathbf{x}) = \int_0^1 \tau\,\mathbf{V}(\mathbf{x}\,\tau)\,d\tau = \int_0^1 \tau\,(0,\cos(y\,\tau),0)\,d\tau.$$

If $y=0$, then

$$\mathbf{S}(x,0,z) = \int_0^1 \tau\,(0,1,0)\,d\tau = \frac{1}{2}\,(0,1,0).$$

If $y \neq 0$, then

$$\int_0^1 \tau\,\cos(y\tau)\,d\tau = \frac{\sin y}{y} + \frac{1}{y^2}\,(\cos y - 1),$$

hence

$$\mathbf{S}(\mathbf{x}) = \begin{cases} \dfrac{1}{2}\,(0,1,0), & \text{for } y=0, \\[2mm] \dfrac{y\sin y + \cos y - 1}{y^2}\,(0,1,0), & \text{for } y \neq 0. \end{cases}$$

Since the case $y=0$ is obtained by taking the limit of the case $y \neq 0$, it suffices in the following only to consider $y \neq 0$. It follows from

$$-\mathbf{x} \times (0,1,0) = (0,1,0) \times \mathbf{x} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 1 & 0 \\ -x & y & z \end{vmatrix} = (z,0,-x),$$

that

$$\begin{aligned} \mathbf{U}(\mathbf{x}) &= -\mathbf{x} \times \mathbf{S}(\mathbf{x}) = \frac{y\sin y + \cos y - 1}{y^2}\,(z,0,-x) \\ &= \left(\frac{\sin y}{y} - \frac{1-\cos y}{y^2}\right)(z,0,-x)(z\,\varphi(y=,0,-x\,\varphi(y)), \end{aligned}$$

where we have put

$$\varphi(y) = \frac{\sin y}{y} - \frac{1-\cos y}{y^2}.$$

First calculate for $y \neq 0$,

$$\begin{aligned} \varphi'(y) &= \frac{\cos y}{y} - \frac{\sin y}{y^2} - \frac{\sin y}{y^2} + 2\cdot\frac{1-\cos y}{y^3} = \frac{\cos y}{y} - 2\cdot\frac{\sin y}{y^2} + 2\cdot\frac{1-\cos y}{y^3} \\ &= \frac{y^2\cos y - 2y\,\sin y + 2 - 2\,\cos y}{y^3}, \end{aligned}$$

where

$$\varphi'(0) = \lim_{y \to 0} \varphi'(0) = 0.$$

Then

$$\mathbf{W}(\mathbf{x}) = \bigtriangledown \times \mathbf{U}(\mathbf{x} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\,\varphi(y) & 0 & -x\,\varphi(y) \end{vmatrix} = (-x\,\varphi'(y), \varphi(y)+\varphi(y), -z\,\varphi'(y))$$

$$= (-x\,\varphi'(y), 2\varphi(y), -z\,\varphi'(y)),$$

which is different from $\mathbf{V}(x,y,z)$.

Example 2.5 *Let $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ be a divergence free vector field. Show that the vector field*

$$\mathbf{W}(x,y,z) = \begin{pmatrix} -\int_\beta^y V_z(x,\eta,\gamma)\,d\eta + \int_\gamma^z V_y(x,y,\zeta)\,d\zeta \\ -\int_\gamma^z V_x(x,y,\zeta)\,d\zeta \\ 0 \end{pmatrix},$$

where β and γ are constants, is a vector potential for $\mathbf{V}$.

A Vector potential.

D Just test the given solution, i.e. show that $\bigtriangledown \times \mathbf{W} = \mathbf{V}$.

I Put $\mathbf{W} = (W_1, W_2, W_3)$. Then

$$\bigtriangledown \times \mathbf{W} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ W_1 & W_2 & W_3 \end{vmatrix} = \left(\frac{\partial W_3}{\partial y} - \frac{\partial W_2}{\partial z}, \frac{\partial W_1}{\partial z} - \frac{\partial W_3}{\partial x}, \frac{\partial W_2}{\partial x} - \frac{\partial W_1}{\partial y} \right).$$

Now,

$$W_1(x,y,z) = -\int_\beta^y V_x(x,\eta,\gamma)\,d\eta + \int_\gamma^z V_y(x,y,\zeta)\,d\zeta,$$

$$W_2(x,y,z) = -\int_\gamma^z V_x(x,y,\zeta)\,d\zeta,$$

and $W_3(x,y,z) = 0$, hence the first coordinate is

$$\frac{\partial W_3}{\partial y} - \frac{\partial W_2}{\partial z} = 0 + \frac{\partial}{\partial z} \int_\gamma^z V_x(x,y,\zeta)\,d\zeta = V_x(x,y,z),$$

and the second coordinate is

$$\frac{\partial W_1}{\partial z} - \frac{\partial W_3}{\partial x} = -\frac{\partial}{\partial z} \int_\beta^y V_z(x,\eta,\gamma)\,d\eta + \frac{\partial}{\partial z} \int_\gamma^z V_y(x,y,\zeta)\,d\zeta - 0$$

$$= 0 + V_y(x,y,z) = V_y(x,y,z).$$

Finally, we get for the third coordinate,

$$\begin{aligned}\frac{\partial W_2}{\partial x}-\frac{\partial W_1}{\partial y} &= -\int_\gamma^z \frac{\partial V_x}{\partial x}(x,y,\zeta)\,d\zeta+V_z(x,y,\gamma)-\int_\gamma^z \frac{\partial V_y}{\partial y}(x,y,\zeta)\,d\zeta \\ &= V_z(x,y,\gamma)-\int_\gamma^z\left\{\frac{\partial V_x}{\partial x}(x,y,\zeta)+\frac{\partial V_y}{\partial y}(x,y,\zeta)\right\}d\zeta.\end{aligned}$$

From the assumption

$$\operatorname{div}\mathbf{V}=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=0,$$

follows by a rearrangement that the integrand is given by

$$-\frac{\partial V_x}{\partial x}-\frac{\partial V_y}{\partial y}=\frac{\partial V_z}{\partial z}.$$

Thus by insertion,

$$\frac{\partial W_2}{\partial x}-\frac{\partial W_1}{\partial y}=V_z(x,y,z)+\int_\gamma^z \frac{\partial V_z}{\partial z}(x,y,\zeta)\,d\zeta=V_z(x,y,\gamma)+\left[V_z(x,y,\zeta)\right]_{\zeta=\gamma}^z=V_z(x,y,z).$$

Summarizing,

$$\bigtriangledown\times\mathbf{W}=\mathbf{V},$$

and we have proved that $\mathbf{W}$ is a vector potential for $\mathbf{V}$.

Remark. The formula of this example of a vector potential in $\mathbb{R}^3$ is far easier to apply than the usual procedure of solution given in most textbooks. $\Diamond$

Example 2.6 *Given the vector field*

$$\mathbf{V}(x,y,z)=\left(2x+x^2y,y-xy^2,7z+5z^3\right),\qquad (x,y,z)\in\mathbb{R}^3.$$

1. *Compute the divergence* $\bigtriangledown\cdot\mathbf{V}$ *and the rotation* $\bigtriangledown\times\mathbf{V}$.

2. *Check if there exists a vector field* $\mathbf{W}:\mathbb{R}^3\to\mathbb{R}^3$, *such that* $\mathbf{V}=\bigtriangledown\times\mathbf{W}$.

Let $L=\{(x,y,z)\in\mathbb{R}^3\mid x\ge 0,\ y\ge 0,\ x^2+y^2+z^2\le 9\}$.

3. *Find the flux of* $\mathbf{V}$ *through* ∂L.

Let $\mathcal{C}$ *denote the closed curve which is the intersection curve of* ∂L *and the plane* $z=0$.

4. *Find the absolute value of the circulation* $\oint_{\mathcal{C}}\mathbf{V}\cdot\mathbf{t}\,ds$.

A Divergence, rotation, vector potential, flux, circulation.

D Follow the guidelines. Apply Gauß's theorem and Stokes's theorem.

I 1) We first get by straightforward calculations,

$$\nabla \cdot \mathbf{V} = \text{ div } \mathbf{V} = (2+2xy)+(1-2xy)+(7+15z^2) = 10+15z^2$$

and

$$\nabla \times \mathbf{V} = \mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x+x^2y & y-xy^2 & 7z+5z^3 \end{vmatrix} = -\left(0,0,x^2+y^2\right).$$

2) Since div $\mathbf{V} \neq \mathbf{0}$, there does not exist a vector field $\mathbf{W}$, such that $\mathbf{V} = \nabla \times \mathbf{W}$, because $\nabla \cdot (\nabla \times \mathbf{V}) = 0$.

3) It follows from Gauss's theorem and 1) that

$$\begin{aligned}
\text{flux}(\partial L) &= \int_{\partial L} \mathbf{V}\cdot\mathbf{n}\,dS = \int_L \operatorname{div}\mathbf{V}\,d\Omega = \int_L (10+15z^2)\,d\Omega \\
&= 10\,\text{vol}(L) + 15\int_L z^2\,d\Omega = 10\cdot\frac{4\pi}{3}\cdot 3^3\cdot\frac{1}{4} + 15\int_{-3}^{3} z^2\cdot\frac{\pi}{4}]\,(9-z^2)\,dz \\
&= 90\pi + \frac{15\pi}{4}\cdot 2\int_0^3 (9z^2 - z^4)\,dz = 90\pi + \frac{15\pi}{2}\left[3z^3 - \frac{1}{5}\,z^5\right]_0^3 \\
&= 90\pi + \frac{15\pi}{2}\left(3^4 - \frac{1}{5}\cdot 3^5\right) = 90\pi + \frac{15\pi}{2\cdot 5}\cdot 3^4\,(5-3) \\
&= 90\pi + 243\pi = 333\pi.
\end{aligned}$$

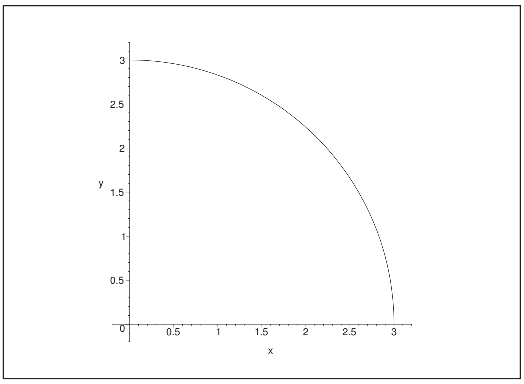


Figure 1: The curve $\mathcal{C}$ and the quarter disc B inside.

4) The curve $\mathcal{C}$ encircles the quarter disc B in the first quadrant of centrum $(0,0)$ and radius 3. Then by Stokes's theorem and 1),

$$\begin{aligned}
\left|\oint_{\mathcal{C}} \mathbf{V}\cdot\mathbf{t}\,ds\right| &= \left|\int_B \mathbf{rot}\;\mathbf{V}\cdot\mathbf{n}\,dx\,dy\right| = \left|-\int_B (0,0,x^2+y^2)\cdot(0,0,1)\,dx\,dy\right| \\
&= \int_B \left(x^2+y^2\right)\,dx\,dy = \int_0^{\frac{\pi}{2}}\left\{\int_0^3 \varrho^2\cdot\varrho\,d\varrho\right\}d\varphi = \frac{\pi}{2}\cdot\left[\frac{\varrho^4}{4}\right]_0^3 = \frac{81\pi}{8}.
\end{aligned}$$

Example 2.7 *A surface of revolution $\mathcal{O}$ with the Z-axis as rotation axis is given in semi polar coordinates (ϱ, φ, z) by*

$$0 \le \varphi \le 2\pi, \qquad 0 \le \varrho \le a \quad og \quad z = a - \frac{\varrho^3}{a^2},$$

where $a \in \mathbb{R}_+$ is a given constant. The surface $\mathcal{O}$ is oriented, such that its unit normal vector $\mathbf{n}$ always has a negative z-coordinate.

1. *Sketch the meridian curve $\mathcal{M}$ of the surface.*

2. *Compute the surface integral*

$$\int_{\mathcal{O}} \left(\frac{a-z}{a}\right)^{\frac{2}{3}} dS.$$

Furthermore, let there be given the vector fields

$$\mathbf{V}(x,y,z) = \left(\frac{y^2}{a^2+z^2} - 1, 1 - \frac{x^2}{a^2+z^2}, 1\right), \qquad (x,y,z) \in \mathbb{R}^3,$$

and

$$\mathbf{U}(x,y,z) = \left(3z - y, 2x + 3z, \frac{x^3+y^3}{a^2+z^2}\right), \qquad (x,y,z) \in \mathbb{R}^3.$$

3. *Prove the existence of a constant $\beta \in \mathbb{R}$, such that*

$$\mathbf{V} = \beta \bigtriangledown \times \mathbf{U},$$

and find β.

4. *Find the flux*

$$\int_{\mathcal{O}} \mathbf{V} \cdot \mathbf{n}\, dS.$$

5. *Find a vector potential for $\mathbf{V}$.*

A Meridian curve; surface integral; flux; vector potential.

D There are many variants of calculations in this example.

I 1) The equation of the meridian curve is

$$z = a - \frac{\varrho^{3\,2}}{a} = a\left\{1 - \left(\frac{\varrho}{a}\right)^3\right\}, \qquad \varrho \in [0, a].$$

2) We can compute the surface integral in several ways.

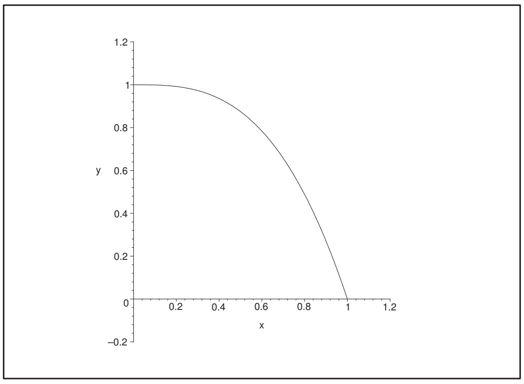


Figure 2: The meridian curve $\mathcal{M}$ for $a = 1$.

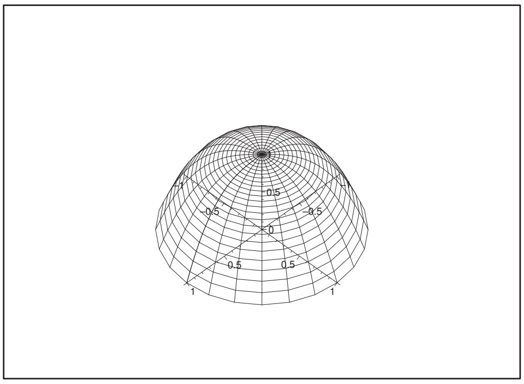


Figure 3: The surface $\mathcal{O}$ for $a = 1$.

a) When we use the reduction theorem of surface integrals we get

$$(3)\quad \int_{\mathcal{O}} \left(\frac{a-z}{a}\right)^{\frac{2}{3}} dS = \int_E \left\{\frac{\varrho(\varphi,t)}{a}\right\}^2 \|\mathbf{N}(\varphi,t)\|\, d\varphi\, dt,$$

where we have used that

$$\frac{a-z}{a} = 1 - \left\{1 - \left(\frac{\varrho}{a}\right)^3\right\} = \left(\frac{\varrho}{a}\right)^3.$$

If we apply the parametric description

$$P(t) = t, \qquad Z(t) = a - \frac{1}{a^2}\, t^3, \qquad t \in [0,a],$$

we get

$$\mathbf{N}(t,\varphi) = P(t)\cdot(-Z'(t)\cos\varphi, -Z'(t)\sin\varphi, P'(t)) = t\left(\frac{3}{a^2}\, t^2\cos\varphi, \frac{3}{a^2}\, t^2\sin\varphi, 1\right),$$

hence,

$$\|\mathbf{N}(t,\varphi)\| = t\sqrt{1+\frac{9}{a^4}\, t^4}, \qquad t \in [0,a], \quad \varphi \in [0,2\pi].$$

Then by insertion into (3),

$$\int_{\mathcal{O}} \left\{ \frac{a-z}{a} \right\}^{\frac{2}{3}} dS = \int_0^1 \left\{ \int_0^{2\pi} \frac{t^2}{a^2} \cdot t \sqrt{1+9\left(\frac{t}{a}\right)^4}\, d\varphi \right\} dt$$

$$= 2\pi \cdot \frac{a^2}{4} \int_0^a \left\{ 1+9\left(\frac{t}{a}\right)^4 \right\}^{\frac{1}{2}} \cdot \frac{4t^3}{a^4}\, dt = \frac{\pi a^2}{18} \int_{t=0}^a \left\{ 1+9\left(\frac{t}{a}\right)^4 \right\}^{\frac{1}{2}} d\left(1+9\left(\frac{t}{a}\right)^4\right)$$

$$= \frac{\pi a^2}{18} \cdot \frac{2}{3} \left[\left(1+9\left(\frac{t}{a}\right)^4 \right)^{\frac{3}{2}} \right]_{t=0}^a = \frac{\pi a^2}{27} \left\{ 10\sqrt{10}-1 \right\}.$$

b) ALTERNATIVELY insert directly into a standard formula:

$$\begin{aligned} \int_{\mathcal{O}} \left(\frac{a-z}{a} \right)^{\frac{2}{3}} dS &= \int_{\mathcal{M}} 2\pi \left(\frac{a-z(\varrho)}{a} \right)^{\frac{2}{3}} \varrho\, ds = 2\pi \int_{\mathcal{M}} \left(\frac{\varrho}{a} \right)^2 \varrho\, ds \\ &= 2\pi a \int_0^a \left(\frac{\varrho}{a} \right)^3 \sqrt{1+9\left(\frac{\varrho}{a}\right)^4}\, d\varrho = \frac{2\pi a^2}{36} \int_0^1 \{1+9t\}^{\frac{1}{2}} d(1+9t) \\ &= \frac{2\pi a^2}{36} \cdot \frac{2}{3} \left[(1+9t)^{\frac{3}{2}} \right]_0^1 = \frac{\pi a^2}{27} \left\{ 10\sqrt{10}-1 \right\}. \end{aligned}$$

3) Clearly, $\mathbf{V}$ is divergence free.
Then by a straightforward calculation,

$$\nabla \times \mathbf{U} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3z-y & 2x+3z & \frac{x^3+y^3}{a^2+z^2} \end{vmatrix} = \left(\frac{3y^2}{a^2+z^2} - 3, 3 - \frac{3x^2}{a^2+z^2}, 2+1 \right) = 3\mathbf{V},$$

hence

$$\mathbf{V} = \frac{1}{3} \nabla \times \mathbf{U},$$

so $\beta = \dfrac{1}{3}$, and $\dfrac{1}{3}\,\mathbf{U}$ is a vector potential for $\mathbf{V}$, cf. 5).

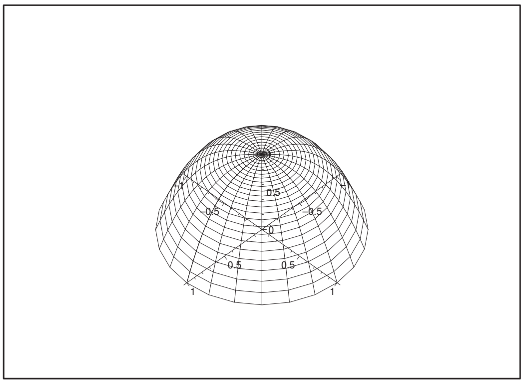


Figure 4: The body Ω for $a = 1$.

4) a) Let$B(\mathbf{0},a)$ denote the disc in the (X,Y)-plane of centrum $(0,0)$ and radius A. The union of the surfaces $\mathcal{O}$ and $B(\mathbf{0},a)$ surrounds a simple body Ω. Since $\mathbf{V}$ is divergence fret, the ingoing flux through $\mathcal{O}$ must be equal to the outgoing flux through $B(\mathbf{0},a)$, where $\mathbf{n} = (0,0,-1)$, hence the flux is

$$\begin{aligned} \int_{\mathcal{O}} \mathbf{V} \cdot \mathbf{n}\, dS &= \int_{B(\mathbf{0},a)} \mathbf{V} \cdot \mathbf{n}\, dS = \int_{B(\mathbf{0}),a)} \left(\frac{{y^2}^2}{a} - 1, 1 - \frac{x^2}{a^2}, 1 \right) \cdot (0,0,-1)\, dS \\ &= -\int_{B(\mathbf{0},a)} dS = -\text{ areal } B(\mathbf{0},a) = -\pi a^2. \end{aligned}$$

b) Alternatively it follows from 3) and Stokes's theorem that

$$\int_{\mathcal{O}} \mathbf{V} \cdot \mathbf{n}\, dS = \frac{1}{3} \int_{\mathcal{O}} (\nabla \times \mathbf{U}) \cdot \mathbf{n}\, dS = \frac{1}{3} \oint_{\partial \mathcal{O}} \mathbf{U} \cdot \mathbf{t}\, ds = \int_{B(\mathbf{0},a)} \mathbf{V} \cdot \mathbf{n}\, dS = \cdots = -\pi a^2,$$

where the dots indicate that we proceed as above.

c) Alternatively we compute the line integral $\dfrac{1}{3} \oint_{\partial \mathcal{O}} \mathbf{U} \cdot \mathbf{t}\, ds$. Here $\partial \mathcal{O}$ is the circle $\varrho = a$ in the plane $z = 0$ run through in a negative sense, because $\mathbf{n}$ has a negative z-component on $\mathcal{O}$. Thus a parametric description of $\partial\mathcal{O}$ is

$$(x,y,z)=a(\cos\varphi,-\sin\varphi,0),\qquad \varphi\in[0,2\pi]$$

where

$$\mathbf{t}=-(\sin\varphi,\cos\varphi,0),\qquad ds=a\,d\varphi,$$

thus

$$\begin{aligned}\int_{\mathcal{O}}\mathbf{V}\cdot\mathbf{n}\,dS &= \frac{1}{3}\oint_{\partial\mathcal{O}}\mathbf{U}\cdot\mathbf{t}\,ds=\frac{1}{3}\oint_{\partial\mathcal{O}}\left(3z-y,2x+3z,\frac{x^3+y^3}{a^2+z^2}\right)\cdot\mathbf{t}\,ds\\
&= -\frac{a}{3}\int_0^{2\pi}\left(\sin\varphi,2\cos\varphi,\cos^3\varphi-\sin^3\varphi\right)\cdot(\sin\varphi,\cos\varphi,0)a\,d\varphi\\
&= -\frac{a^2}{3}\int_0^{2\pi}\{\sin^2\varphi+2\cos^2\varphi\}d\varphi=-\frac{a^2}{3}\left(\frac{3}{2}\cdot 2\pi\right)=-\pi a^2.\end{aligned}$$

d) Alternatively there are also variants in which Green's theorem in the plane occurs. We shall only demonstrate one of them;

$$\begin{aligned}\int_{\mathcal{O}}\mathbf{V}\cdot\mathbf{n}\,dS &= \frac{1}{3}\oint_{\partial\mathcal{O}}\mathbf{t}\cdot\mathbf{U}\,ds=-\frac{1}{3}\int_{B(\mathbf{0},a)}\left(\frac{\partial U_y}{\partial x}-\frac{\partial U_x}{\partial y}\right)dS\\
&= -\frac{1}{3}\int_{B(\mathbf{0},a)}(2+1)\,dS=-\text{ areal }B(\mathbf{0},a)=-\pi a^2.\end{aligned}$$

5) Now, $\mathbf{W}_0$ is a vector potential for $\mathbf{V}$, if $\mathbf{V}=\bigtriangledown\times\mathbf{W}_0$. This is according to 3) fulfilled for

$$\mathbf{W}_0=\frac{1}{3}\,\mathbf{U}=\frac{1}{3}\left(3z-y,2x+3z,\frac{x^3+y^3}{a^2+z^2}\right).$$

Alternatively (and far more difficult) we can find a vector potential $\mathbf{W}_0$ directly by means of the standard formula,

$$\mathbf{W}_0(\mathbf{x})=-\mathbf{x}\times\int_0^1\tau\,\mathbf{V}(\tau\,\mathbf{x})\,d\tau.$$

Here

$$\int_0^1\tau\,\mathbf{V}(\tau\,\mathbf{x})\,d\tau=\left(\int_0^1\tau\left\{\frac{\tau^2y^2}{a^2+\tau^2z^2}-1\right\}d\tau,\int_0^1\tau\left\{1-\frac{\tau^2x^2}{a^2+\tau^2z^2}\right\}d\tau,\int_0^1\tau\,d\tau\right).$$

We get by a calculation for $z\neq 0$,

$$\begin{aligned}\int_0^1\tau\cdot\frac{\tau^2y^2}{a^2+\tau^2z^2}\,d\tau &= \frac{y^2}{z^2}\int_0^1\tau\cdot\frac{\tau^2z^2}{a^2+\tau^2z^2}\,d\tau=\frac{y^2}{z^2}\int_0^1\tau\left(1-\frac{1}{1+\dfrac{z^2}{a^2}\,\tau^2}\right)d\tau\\
&= \frac{y^2}{z^2}\left[\frac{\tau^2}{2}-\frac{1}{2}\,\frac{a^2}{z^2}\,\ln\left\{1+\frac{z^2}{a^2}\,\tau^2\right\}\right]_{\tau=0}^1=\frac{y^2}{2z^2}-\frac{a^2y^2}{2z^4}\,\ln\left(1+\frac{z^2}{a^2}\right).\end{aligned}$$

By taking the limit, or by a direct computation, we get

$$\int_0^1\tau\cdot\frac{\tau^2y^2}{a^2+\tau^2z^2}\,d\tau=\frac{y^2}{4a^2}\qquad\text{for } z=0.$$

Similarly

$$\int_0^1 \tau \cdot \frac{\tau^2 x^2}{a^2+\tau^2 z^2}\,d\tau = \frac{x^2}{2z^2} - \frac{a^2x^2}{2z^4}\,\ln\left(1+\frac{z^2}{a^2}\right) \qquad \text{for } z \neq 0,$$

and

$$\int_0^1 \tau \cdot \frac{\tau^2 x^2}{a^2+\tau^2 z^2}\,d\tau = \frac{x^2}{4a^2} \qquad \text{for } z = 0.$$

Due to the continuity it suffices in the following with the expressions for $z \neq 0$. Then

$$\int_0^1 \tau\,\mathbf{V}(\tau\,\mathbf{x})\,d\tau = \frac{1}{2}\begin{pmatrix} -1+\dfrac{y^2}{z^2} - \dfrac{a^2y^2}{z^4}\,\ln\left\{1+\dfrac{z^2}{a^2}\right\} \\[2ex] 1-\dfrac{x^2}{z^2} + \dfrac{a^2x^2}{z^4}\,\ln\left\{1+\dfrac{z^2}{a^2}\right\}, \\[2ex] 1 \end{pmatrix}.$$

We now find $\mathbf{W}_0$ by

$$\begin{aligned}
\mathbf{W}_0(\mathbf{x}) &= \int_0^1 \tau\,\mathbf{V}(\tau\,\mathbf{x})\,d\tau \times \mathbf{x} \\
&= \frac{1}{2}\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -1+\dfrac{y^2}{z^2}-\dfrac{a^2y^2}{z^4}\ln\left(1+\dfrac{z^2}{a^2}\right) & 1-\dfrac{x^2}{z^2}+\dfrac{a^2x^2}{z^4}\ln\left(1+\dfrac{z^2}{a}\right) & 1 \\ x & y & z \end{vmatrix} \\
&= \frac{1}{2}\begin{pmatrix} z-y-\dfrac{x^2}{z}+\dfrac{a^2x^2}{z^3}\,\ln\left(1+\dfrac{z^2}{a^2}\right) \\[2ex] x+z-\dfrac{y^2}{z}+\dfrac{a^2y^2}{z^3}\,\ln\left(1+\dfrac{z^2}{a^2}\right) \\[2ex] -x-y+\dfrac{y^3+x^3}{z^2}-\dfrac{a^2}{z^4}\,(x^3+y^3)\ln\left(1+\dfrac{z^2}{a^2}\right) \end{pmatrix}, \qquad z \neq 0.
\end{aligned}$$

For $z = 0$ the result is obtained by taking the limit.

This horrible expression is of course not equal to $\dfrac{1}{3}\,\mathbf{U}$. On the other hand, a vector potential is not unique. Here we can only check our computations by insertion.

C TEST. Put

$$\mathbf{W}_0 = (W_1, W_2, W_3) \quad \text{and} \quad \mathbf{V} = (V_1, V_2, V_3).$$

Then

$$
\begin{aligned}
\frac{\partial W_3}{\partial y}-\frac{\partial W_2}{\partial z} &= \frac{1}{2}\left\{-1+\frac{3y^2}{z^2}-\frac{3a^2y^2}{z^4}\ln\left(1+\frac{z^2}{a^2}\right)-1-\frac{y^2}{z^2}\right.\\
&\qquad\left.+\frac{3a^2y^2}{z^4}\ln\left(1+\frac{z^2}{a^2}\right)-\frac{a^2y^2}{z^3}\cdot\frac{2z}{1+\dfrac{z^2}{a^2}}\cdot\frac{1}{a^2}\right\}\\
&= \frac{1}{2}\left\{-2+\frac{2y^2}{z^2}-\frac{a^2y^2}{z^2}\cdot\frac{2}{a^2+z^2}\right\}\\
&= \frac{1}{2}\left\{-2+\frac{2y^2}{z^2(a^2+z^2)}\left(a^2+z^2-a^2\right)\right\}=V_1.
\end{aligned}
$$

The computation of $\dfrac{\partial W_1}{\partial z}-\dfrac{\partial W_3}{\partial x}=V_2$ is similar, where we could apply the "asymmetry" (x and y are interchanged and we also change sign). Finally,

$$\frac{\partial W_2}{\partial x}-\frac{\partial W_1}{\partial y}=\frac{1}{2}\{1+1\}=1=V_3,$$

hence the found vector field $\mathbf{W}_0(\mathbf{x})$ is a vector potential for $\mathbf{V}$.

Example 2.8 *1) Find the divergence of the vector field*

$$\mathbf{V}(x,y,z) = (\sin y + \cos z, \sin z + \cos x, \sin x + \cos y), \qquad (x,y,z) \in \mathbb{R}^3.$$

2) Prove the existence of a constant α, such that **rot V** $= \alpha\mathbf{V}$*. Then find a vector potential for* $\mathbf{V}$*.*

A Divergence, rotation and vector potential.

D Just compute. In 2) one might get a better solution.

I 1) Clearly,

$$\text{div}\,\mathbf{V} = 0.$$

Then compute

$$\begin{aligned} \nabla \times \mathbf{v} = \mathbf{rot}\ \mathbf{V} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin y + \cos z & \sin z + \cos x & \sin x + \cos y \end{vmatrix} \\ &= (-\sin y - \cos z, -\sin z - \cos x, -\sin x - \cos y) = -\mathbf{V}(x,y,z). \end{aligned}$$

2) It follows immediately that $\nabla \times (-\mathbf{V}) = \mathbf{V}$, thus $-\mathbf{V}$ is according to the definition a vector potential for $\mathbf{V}$.

Alternatively, $\mathbf{V}$ is divergence free, thus there exists a vector potential. One of these is given by

$$\mathbf{W} = -\mathbf{x} \times \mathbf{S}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \times \mathbf{x},$$

where

$$\mathbf{S}(\mathbf{x}) = \int_0^1 \tau\,\mathbf{V}(\tau\,\mathbf{x})\,d\tau = \begin{pmatrix} \int_0^1 \tau\{\sin(\tau y) + \cos(\tau z)\}\,d\tau \\ \int_0^1 \tau\{\sin(\tau z) + \cos(\tau x)\}\,d\tau \\ \int_0^1 \tau\{\sin(\tau x) + \cos(\tau y)\}\,d\tau \end{pmatrix}.$$

By some small calculations we get

$$\int_0^1 \tau\,\sin\tau v\,d\tau = \begin{cases} \dfrac{1}{v^2}\{\sin v - v\cos v\} & \text{for } v \neq 0, \\ 0 & \text{for } v = 0, \end{cases}$$

and

$$\int_0^1 \tau\,\cos\tau v\,d\tau = \begin{cases} \dfrac{1}{v^2}\{\cos v - 1 + v\,\sin v\} & \text{for } v \neq 0, \\ \dfrac{1}{2} & \text{for } v = 0. \end{cases}$$

By insertion of these expressions into $\mathbf{S}(\mathbf{x})$ we get the rather complicated vector potential

$$\begin{aligned}\mathbf{W}(\mathbf{x}) &= \mathbf{S}(\mathbf{x}) \times \mathbf{x} \\ &= \begin{pmatrix} \dfrac{1}{z}\{\sin z - z\cos z\} + \dfrac{z}{x^2}\{\cos x - 1 + x\sin x\} \\[2mm] \dfrac{1}{x}\{\sin x - x\cos x\} + \dfrac{x}{y^2}\{\cos y - 1 + y\sin y\} \\[2mm] \dfrac{1}{y}\{\sin y - y\cos y\} + \dfrac{y}{z^2}\{\cos z - 1 + z\sin z\} \end{pmatrix} \\ &\qquad - \begin{pmatrix} \dfrac{y}{x^2}\{\sin x - x\cos x\} + \dfrac{1}{y}\{\cos y - 1 + y\sin y\} \\[2mm] \dfrac{z}{y^2}\{\sin y - y\cos y\} + \dfrac{1}{z}\{\cos z - 1 + z\sin z\} \\[2mm] \dfrac{x}{z^2}\{\sin z - z\cos z\} + \dfrac{1}{x}\{\cos x - 1 + x\sin x\} \end{pmatrix},\end{aligned}$$

with suitable interpretations when x, y or $z = 0$.

Example 2.9 *Consider the vector field*

$$\mathbf{V}(x,y,z) = (2x+3y, 2y+3x, -4z), \qquad (x,y,z) \in \mathbb{R}^3,$$

and the function

$$G(x,y,z) = \alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy, \qquad (x,y,z) \in \mathbb{R}^3,$$

where α, β, γ, δ *are constants.*

1. *Show that one can choose the constants* α, β, γ, δ *such that* $\mathbf{V} = \bigtriangledown G$.

2. *Compute the tangential line integral*

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, ds,$$

where $\mathcal{K}$ *is the broken line composed of the two line segments from* $(2a,0,a)$ *via* $(2a,0,0)$ *to* $(a,0,0)$.

3. *Show that the vector field*

$$\mathbf{W}(x,y,z) = \left(2yz + xz - x^2, -2xz - yz + y^2, y^2 - x^2 + z^2\right), \qquad (x,y,z) \in \mathbb{R}^3,$$

is a vector potential for $\mathbf{V}$.

Let a *be a positive constant, and let* $\mathcal{F}$ *be the oriented surface given by*

$$x^2+y^2+z^2 = a^2, \qquad z \geq 0,$$

with the unit normal vector $\mathbf{n}$ *pointing away from* $(0,0,0)$.

4. *Find the flux*

$$\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n}\, dS.$$

A Vector analysis, i.e. check the gradient field, tangential line integral, vector potential and flux.

D The examples can be solved in many ways, and I have probably not found all variants. Below we give the following variants:

1) We solve 1) in 5 variants.
2) We solve 2) in 2 variants.
3) We solve 3) in 2 variants.
4) We solve 4) in 4 variants and 1 subvariants (and there are more; we miss e.g. the calculations when $\mathcal{F}$ is a surface of revolution).

I 1) **First variant.** *A simple test.*
When we compute $\bigtriangledown G$ we get

$$\bigtriangledown G = (2\alpha x + \delta y, \delta x + 2\beta y, 2\gamma z).$$

Choose $\alpha = 1$, $\beta = 1$, $\gamma = -2$ and $\delta = 3$. Then

$$\bigtriangledown G = (2x+3y, 3y+2y, -4z) = \mathbf{V},$$

and $\mathbf{V} = \bigtriangledown G$ is a gradient field with the integral

$$G(x,y,z) = x^2+y^2-2z^2+3xy.$$

Second variant. *Manipulation.*

We conclude from

$$\begin{aligned}\mathbf{V}\cdot d\mathbf{x} &= (2x+3y)dx+(2y+3x)dy-4z\,dz\\ &= d\left(x^2\right)+d\left(y^2\right)-d\left(2z^2\right)+3\{y\,dx+x\,dy\}\\ &= d\left(x^2+y^2-2z^2+3xy\right)\end{aligned}$$

that

$$G(x,y)=x^2+y^2-2z^2+3xy$$

is an integral of $\bigtriangledown G=\mathbf{V}$, and $\mathbf{V}$ is a gradient field.

Then by comparison, $\alpha=1$, $\beta=1$, $\gamma=-2$, $\delta=-3$.

Third variant. *Indefinite integration.*

Put

$$\omega=\mathbf{V}\cdot d\mathbf{x}=(2x+3y)dx+(2y+3x)dy-4z\,dz.$$

Then

$$F_1(x,y,z);=\int(2x+3y)dx=x^2+3xy,$$

thus

$$\omega-dF_1=(2x+3y)dx+(2y+3x)dy-4z\,dz-\{(2x+3y)dx+3x\,dy\}=2y\,dy-4z\,dz,$$

which is reduced to

$$\omega-d\left(x^2+3xy\right)=d\left(y^2-2z^2\right).$$

Then by a rearrangement,

$$\omega=d\left(x^2+3xy\right)+d\left(y^2-2z^2\right)=d\left(x^2+y^2-2z^2+3xy\right),$$

and we conclude that

$$G(x,y,z)=x^2+y^2-2z^2+3xy$$

is an integral of $\mathbf{V}$, i.e. $\mathbf{V}=\bigtriangledown G$, and $\alpha=1$, $\beta=1$, $\gamma=-2$, $\delta=3$.

The latter two variants *assume* that we have proved that $\mathbf{V}$ is a gradient field. First note that

$$\mathbf{rot}\;\mathbf{V}=\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2x+3y & 3x+2y & -4z\end{vmatrix}=(0-0,0-0,3-3)=\mathbf{0},$$

and $\mathbf{V}$ is rotation free. The domain $\mathbb{R}^3$ is simply connected (it is even convex), hence $\mathbf{V}$ is a gradient field.

Fourth variant. *Integration along a broken line.*

We get by a tangential line integration along the broken line

$$(0,0,0) \longrightarrow (x,0,0) \longrightarrow (x,y,0) \longrightarrow (x,y,z) \qquad \text{in } \mathbb{R}^3$$

that

$$\int_{\mathbf{0}}^{\mathbf{x}} \mathbf{V}\cdot d\mathbf{x} = \int_0^x 2t\,dt + \int_0^y (2t+3x)\,dt - \int_0^z 4t\,dt = x^2+y^2+3xy-2z^2.$$

Since we already have proved that $\mathbf{V}$ is a gradient field, an integral is given by

$$G(x,y,z) = x^2+y^2-2z^2+3xy, \qquad \bigtriangledown G = \mathbf{V},$$

and we get by comparison that $\alpha = 1$, $\beta = 1$, $\gamma = -2$, $\delta = 3$.

Fifth variant. *Radial integration.*

We have above proved that $\mathbf{V}$ is a gradient field. Therefore,

$$\begin{aligned} G(x,y,z) &= (x,y,z)\cdot\int_0^1 \mathbf{V}(x\tau, y\tau, z\tau)\,d\tau \\ &= (x,y,z)\cdot\int_0^1 ((2x+3y)\tau,(2y+3x)\tau,-4z\tau)\,d\tau \\ &= (x,y,z)\cdot(2x+3y,2y+3x,-4z)\int_0^1 \tau\,d\tau \\ &= \frac{1}{2}\left\{(2x^2+3xy)+(2y^2+3xy)-4z^2\right\} \\ &= x^2+y^2-2z^2+3xy \end{aligned}$$

is an integral of $\mathbf{V}$, i.e. $\bigtriangledown G = \mathbf{V}$, and we get by comparison that $\alpha = 1$, $\beta = 1$, $\gamma = -2$, $\delta = 3$.

2) **First variant.** *The gradient theorem.*

According to 1), the field $\mathbf{V}$ is a gradient field with the integral

$$G(x,y,z) = x^2+y^2-2z^2+3xy.$$

Then by the gradient theorem,

$$\int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,ds = G(a,0,0) - G(2a,0,a) = a^2 - (4a^2-2a^2) = -a^2.$$

Second variant. *Line integral.*

We have on the line segment from $(2a,0,a)$ to $(2a,0,0)$ that $x = 2a$ and $y = 0$, while z runs through the interval $[0,a]$ from a to 0 (the reverse direction).

On the line segment from $(2a,0,0)$ to $(a,0,0)$, the variable x runs through the interval $[a,2a]$ from $2a$ towards a, also in the reverse direction, while $y = 0$ and $z = 0$.

As a conclusion we get

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,ds &= \int_a^0 (-4t)dt + \int_{2a}^a (2t+3\cdot 0)dt = \left[-2t^2\right]_a^0 + \left[t^2\right]_{2a}^a \\ &= 2a^2 + (a^2-4a^2) = -a^2. \end{aligned}$$

3) **First variant.** *Test.*
We shall only prove that $\mathbf{V} = \bigtriangledown \times \mathbf{W}$. We get

$$\begin{aligned}\bigtriangledown \times \mathbf{W} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \\ 2yz+xz-x^2 & -2xz-yz+y^2 & y^2-x^2+z^2 \end{vmatrix} \\ &= (2y-(-2x-y),(2y+x)-(-2x),-2z-2z) \\ \\ &= (3y+2x,3x+2y,-4z) = \mathbf{V},\end{aligned}$$

and $\mathbf{W}$ is a vector potential for $\mathbf{V}$.

Second variant. *Insertion into a standard formula.*
The assumptions are that $\mathbb{R}^3$ is star shaped (obvious) and that $\mathbf{V}$ is divergence free. By a small computation,

$$\operatorname{div} \mathbf{V} = 2 + 2 - 4 = 0,$$

and it follows that $\mathbf{V}$ has a vector potential, which can be found by the formula

$$\mathbf{W}_0(\mathbf{x}) = -\mathbf{x} \times \int_0^1 \tau \, \mathbf{V}(\tau \mathbf{x})\, d\tau = \left\{ \int_0^1 \tau \, \mathbf{V}(\tau \mathbf{x})\, d\tau \right\} \times \mathbf{x}.$$

Since $\mathbf{V}$ is homogeneous of first degree,

$$\mathbf{V}(\tau \mathbf{x}) = \tau \, \mathbf{V}(\mathbf{x}),$$

it follows by insertion that

$$\begin{aligned}
\mathbf{W}_0(\mathbf{x}) &= \left\{ \int_0^1 \tau \cdot \tau \, d\tau \right\} \mathbf{V}(\mathbf{x}) \times \mathbf{x} = \frac{1}{3} \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2x+3y & 2y+3x & -4z \\ x & y & z \end{vmatrix} \\
&= \frac{1}{3} \left(z\{2y+3x\}+4zy, -4zx-z\{2x+3y\}, 2xy+3y^2-2xy-3x^2 \right) \\
&= \frac{1}{3} \left(z\{3x+6y\}, -z\{6x+3y\}, 3y^2-3x^2 \right) \\
&= \left(2yz+xz, -2xz-yz, y^2-x^2 \right) \\
&= \mathbf{W}(\mathbf{x}) + \left(-x^2, y^2, z^2 \right),
\end{aligned}$$

hence

$$\nabla \times \mathbf{W} = \nabla \times \mathbf{W}_0 + \nabla \times \left(-x^2, y^2, z^2 \right) = \mathbf{V} + \mathbf{0} = \mathbf{0}.$$

We conclude that both $\mathbf{W}_0$ and $\mathbf{W}$ are vector potentials for $\mathbf{V}$.

4) **First variant.** *Stokes's theorem.*
When the unit normal vector is pointing away from $(0,0,0)$, we get the natural orientation of the bounding curve (a circle in the XY-plane),

$$\delta\mathcal{F}: \quad \mathbf{r}(t) = a(\cos t, \sin t, 0), \qquad t \in [0, 2\pi],$$

in its positive sense.
It follows from 3) that $\mathbf{V} = \nabla \times \mathbf{W}$, hence the flux is according to Stokes's theorem

$$\begin{aligned}
&\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n}\, dS = \int_{\mathcal{F}} (\nabla \times \mathbf{W}) \cdot \mathbf{n}\, dS = \int_{\delta\mathcal{F}} \mathbf{W} \cdot \mathbf{t}\, ds \\
&\quad = \int_0^{2\pi} (0-a^2\cos^2 t, 0-a^2\sin^2, a^2\sin^2 t - a^2\cos^2 t) \cdot a(-\sin t, \cos t, 0)\, dt \\
&\quad = a^3 \int_0^{2\pi} \{\cos^2 t\, \sin t - \sin^2 t\, \cos t\}\, dt = a^3 \left[-\frac{\cos^3 t}{3} - \frac{\sin^3 t}{3} \right]_0^{2\pi} = 0.
\end{aligned}$$

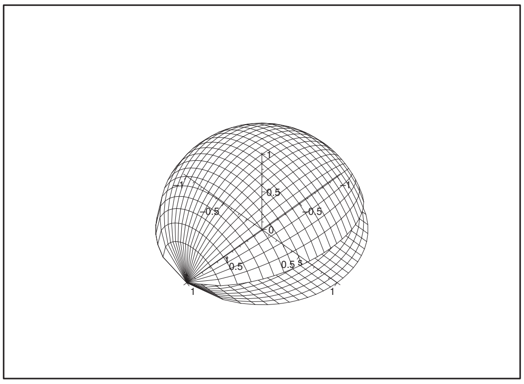


Figure 5: The half sphere $\mathcal{F}$ and the "bounding curve" $\delta\mathcal{F}$ for $a = 1$.

Subvariant. From the second variant of 3) we also obtain that $\mathbf{V} = \bigtriangledown \times \mathbf{W}_0$. Now, $\mathbf{W}_0 = (0, 0, \cdots)$ and $\mathbf{t} = (\cdots, \cdots, 0)$ on $\delta\mathcal{F}$, so an application of Stokes's theorem shows that the flux is

$$\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n}\, dS = \int_{\mathcal{F}} (\bigtriangledown \times \mathbf{W}_0) \cdot \mathbf{n}\, dS = \int_{\delta\mathcal{F}} \mathbf{W}_0 \cdot \mathbf{t}\, ds = \int_{\delta\mathcal{F}} 0\, ds = 0.$$

Second variant. *Surface integral, rectangular coordinates.*

The unit normal vector is

$$\mathbf{n} = \frac{1}{a}(x, y, z), \qquad \text{on } \mathcal{F},$$

hence the flux is

$$\begin{aligned}\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n}\, dS &= \int_{\mathcal{F}} \frac{1}{a}(2x+3y, 3x+2y, -4z) \cdot (x, y, z)\, dS = \frac{1}{a}\int_{\mathcal{F}} \left(2x^2+3xy+2y^2-4z^2\right) dS \\ &= \frac{2}{a}\int_{\mathcal{F}} \left(x^2 + y^2 - 2z^2 + \frac{3}{2}\, xy\right) dS = \frac{2}{a}\int_{\mathcal{F}} (x^2 + y^2 - 2z^2)\, dS,\end{aligned}$$

because $\int_{\mathcal{F}} xy\, dS = 0$ of symmetric reasons.

It also follows by the symmetry that

$$\int_{\mathcal{F}} x^2\, dS = \int_{\mathcal{F}} y^2\, dS.$$

Let $\mathcal{F}_1$ be given by

$$x^2 + y^2 + z^2 = a^2, \qquad y \geq 0.$$

Then we get in exactly the same way,

$$\int_{\mathcal{F}} x^2\, dS = \int_{\mathcal{F}_1} x^2\, dS = \int_{\mathcal{F}_1} z^2\, dS = \int_{\mathcal{F}} z^2\, dS,$$

thus

$$\int_{\mathcal{F}} x^2\, dS = \int_{\mathcal{F}} y^2\, dS = \int_{\mathcal{F}} z^2\, dS.$$

Hence by insertion,

$$\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,dS = \frac{2}{a}\left\{\int_{\mathcal{F}} x^2\,dS + \int_{\mathcal{F}} y^2\,dS - 2\int_{\mathcal{F}} z^2\,dS\right\} = 0.$$

Third variant. *Surface integral, spherical coordinates.*
In spherical coordinates a parametric description of the surface is given by

$$\begin{cases} x = a\,\sin\theta\,\cos\varphi, \\ y = a\,\sin\theta\,\sin\varphi, \qquad \theta\in\left[0,\dfrac{\pi}{2}\right], \quad \varphi\in[0,2\pi]. \\ z = a\,\cos\theta, \end{cases}$$

Thus the normal vector becomes

$$\begin{aligned}\mathbf{N}(\theta,\varphi) &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a\,\cos\theta\,\cos\varphi & a\,\cos\theta\,\sin\varphi & -a\,\sin\theta \\ -a\,\sin\theta\,\sin\varphi & a\,\sin\theta\,\cos\varphi & 0 \end{vmatrix} \\ &= a^2\left(\sin^2\theta\,\cos\varphi, \sin^2\theta\,\sin\varphi, \sin\theta\,\cos\theta\right) \\ &= a^2\sin\theta\,(\sin\theta\,\cos\varphi, \sin\theta\,\sin\varphi, \cos\theta),\end{aligned}$$

and we note that the z-component is positive, showing that we have obtained the right orientation. Then

$$\begin{aligned}&\mathbf{V}(\mathbf{x}(\theta,\varphi))\cdot\mathbf{N}(\theta,\varphi) \\ &= (2a\sin\theta\cos\varphi{+}3a\sin\theta\sin\varphi, 3a\sin\theta\cos\varphi{+}2a\sin\theta\sin\varphi, -4a\cos\theta)\cdot \\ &\qquad\cdot(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,a^2\sin\theta \\ &= a^3\left\{2\sin^2\theta\cos^2\varphi + 3\sin^2\theta\sin\varphi\cos\varphi + 3\sin^2\theta\sin\varphi\cos\varphi \right. \\ &\qquad \left. +2\sin^2\theta\sin^2\varphi - 4\cos^2\theta\right\}\sin\theta \\ &= a^3\left\{2\sin^2\theta + 6\sin^2\theta\sin\varphi\cos\varphi - 4\cos^2\theta\right\}\sin\theta \\ &= a^3\left\{2 - 2\cos^2\theta - 4\cos^2\theta + 6\sin^2\theta\sin\varphi\cos\varphi\right\}\sin\theta \\ &= 2a^3\left\{1 - 3\cos^2\theta + 3\sin^2\theta\sin\varphi\cos\varphi\right\}\sin\theta.\end{aligned}$$

The flux is

$$\begin{aligned}\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,dS &= \int_E \mathbf{V}(\mathbf{x}(\theta,\varphi))\cdot\mathbf{N}(\theta,\varphi)\,d\theta\,d\varphi \\ &= 2a^3\int_0^{2\pi}\left\{\int_0^{\frac{\pi}{2}}\left(1 - 3\cos^2\theta + 3\sin^2\theta\sin\varphi\cos\varphi\right)\sin\theta\,d\theta\right\}d\varphi \\ &= 4\pi a^3\int_0^{\frac{\pi}{2}}\left\{1 - 3\cos^2\theta\right\}\sin\theta\,d\theta = 4\pi a^3\left[-\cos\theta + \cos^3\theta\right]_0^{\frac{\pi}{2}} = 0.\end{aligned}$$

Fourth variant. *Gauß's theorem.*

First note that $\mathcal{F}$ does not surround any body Ω, so we cannot apply Gauß's theorem immediately. However, if we add the plane surface ("the bottom")

$$B = \{(x,y,0) \mid x^2+y^2 \le a^2\}$$

with the unit normal vector $\mathbf{n} = (0,0,-1)$, then the union $\mathcal{F} \cup B$ surrounds the half ball Ω. We found above that $\operatorname{div} \mathbf{V} = 0$, so we conclude by Gauß's theorem that

$$\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,dS + \int_B \mathbf{V}\cdot(0,0,-1)\,dS = \int_\Omega \operatorname{div}\,\mathbf{V}\,d\Omega = \int_\Omega 0\,d\Omega = 0,$$

hence by a rearrangement,

$$\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,dS = +\int_B \mathbf{V}\cdot(0,0,1)\,dS = \int_B 0\,dS = 0,$$

where we have used that $\mathbf{V} = (2x+3y, 3x+2y, 0)$ on B.

Fifth variant. *The surface as a surface of revolution.*

According to the second variant we shall compute the surface integral

$$\int_{\mathbf{F}} \mathbf{V}\cdot\mathbf{n}\,dS = \frac{2}{a}\int_{\mathcal{F}} (x^2+y^2-2z^2)\,dS.$$

It can of course be done do by considering $\mathcal{F}$ as a surface of revolution. We shall leave this variant to the reader.

Example 2.10 *Given the two functions*

$$L(x,y) = e^y + y(e^x - e^{xy}), \qquad (x,y) \in \mathbb{R}^2,$$

$$M(x,y) = e^x + x(e^y - e^{xy}), \qquad (x,y) \in \mathbb{R}^2.$$

1) Prove that the vector field $\mathbf{V}(x,y) = (L(x,y), M(x,y))$, $(x,y) \in \mathbb{R}^2$, *is a gradient field, and find all integrals of* $\mathbf{V}$.

2) Show that the vector field

$$\mathbf{U}(x,y,z) = (z\,M(x,y), -z\,L(x,y), L(x,y)+M(x,y)), \qquad (x,y,z) \in \mathbb{R}^3,$$

is not *a gradient field, while there exists a vector potential for* $\mathbf{U}$. *(One shall* not *find such a vector potential).*

A Gradient field, vector potential.

D We shall prove in three ways that $\mathbf{V}$ is a gradient field. That $\mathbf{U}$ has a vector potential is shown by means of the necessary and sufficient conditions.

I 1) First note that $L(x,y)$ and $M(x,y)$ are of class C^∞ in all of $\mathbb{R}^2$.

First method. *Manipulation.*
By means of the rules of calculations we get by some manipulation,

$$\begin{aligned} L\,dx + M\,dy &= \{e^y+y(e^x-e^{xy})\}\,dx + \{e^x+x(e^y-e^{xy})\}\,dy \\ &= \{e^y dx + xe^y dy\} + \{ye^x dx + e^x dy\} - e^{xy}\{y\,dx + x\,dy\} \\ &= \{e^y dx + x d\,(e^y)\} + \{y\,d\,(e^x) + e^x dy\} - e^{xy} d(xy) \\ &= d\,(xe^y) + d\,(ye^x) - d\,(e^{xy}) = d\,(xe^y + ye^x - e^{xy}) \\ &= \bigtriangledown F \cdot (dx, dy). \end{aligned}$$

Hence $(L(x,y), M(x,y))$ is a gradient field and its integrals are given by

$$F(x,y) = xe^y + ye^x - e^{xy} + C, \qquad C \text{ arbitrary constant.}$$

Second method. *Indefinite integration.*
We first get

$$F_1(x,y) = \int L(x,y)\,dx = \int \{e^y + y(e^x - e^{xy})\}\,dx = xe^y + ye^x - e^{xy}.$$

Then by a check

$$\frac{\partial F_1}{\partial y} = xe^y + e^x - xe^{xy} = e^x + x(e^y - e^{xy}) = M(x,y),$$

which shows that $(L(x,y), M(x,y))$ is a gradient field and that its integrals are

$$F(x,y) = F_1(x,y) + C = xe^y + ye^x - e^{xy} + C, \qquad (x,y) \in \mathbb{R}^2,$$

where C is an arbitrary constant.

Third method. *Integration along a broken line followed by a check.*
When we integrate $L\,dx + M\,dy$ along the broken line

$$(0,0) \longrightarrow (x,0) \longrightarrow (x,y),$$

we get the *candidate*

$$\begin{aligned} F(x,y) &= \int_0^x L(t,0)\,dt + \int_0^y M(x,t)\,dt = \int_0^x dt + \int_0^y \left\{e^x + x(e^t - e^{xt})\right\}\,dt \\ &= x + \left[te^x + xe^t - e^{xt}\right]_0^y = x + ye^x + xe^y - e^{xy} - x + 1 \\ &= ye^x + xe^y - e^{xy} + 1. \end{aligned}$$

By *testing* (this is mandatory by this method) we get

$$\frac{\partial F}{\partial x} = ye^x + e^y - ye^{xy} = e^y + y(e^x - e^{xy}) = L(x,y),$$

$$\frac{\partial F}{\partial y} = e^x + xe^y - xe^{xy} = e^x + x(e^y - e^{xy}) = M(x,y).$$

It follows from the above that $(L(x,y), M(x,y))$ is a gradient field and that its integrals are

$$F(x,y) = xe^y + ye^x - e^{xy} + C, \qquad C \text{ a arbitrary constant.}$$

2) Now

$$\frac{\partial}{\partial z}\{z\,M(x,y)\} = M(x,y) = e^x + x(e^y - e^{xy})$$

and

$$\frac{\partial}{\partial x}\{L(x,y) + M(x,y)\} = ye^x - y^2e^{xy} + e^x - e^{xy} - xye^{xy} \neq \frac{\partial}{\partial z}\{z\,M(x,y)\},$$

so the necessary conditions of a gradient field are not satisfied, and $\mathbf{U}$ is not a gradient field.

Clearly, $\mathbf{U}$ is of class C^∞ in all of $\mathbb{R}^3$, and $\mathbb{R}^3$ is star shaped. (It is even convex.)
As (L, M) is a gradient field, we have in particular

$$\frac{\partial L}{\partial y} = \frac{\partial M}{\partial x},$$

thus

$$\operatorname{div} \mathbf{U} = z\,\frac{\partial M}{\partial x} - z\,\frac{\partial L}{\partial y} + 0 = z\left\{\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right\} = 0,$$

and $\mathbf{U}$ is divergence free and defined in a star shaped domain. Therefore $\mathbf{U}$ has a vector potential.

Remark. In principal the integrals of the formula of the vector potential can be computed. However, the result is very difficult to manage with a lot of exceptional cases. For this reason it is highly recommended always to find some other method before one tries to find the vector potential by means of the standard formulæ. $\lozenge$

Example 2.11 *Given the vector field*

$$\mathbf{V}(x,y,z) = (\cos y - \sin z, \cos z - \sin x, \cos x - \sin y), \qquad (x,y,z) \in \mathbb{R}^3.$$

1) Find the divergence $\bigtriangledown \cdot \mathbf{V}$ *and the rotation* $\bigtriangledown \times \mathbf{V}$.

2) Show the existence of a constant α*, such that* $\alpha\mathbf{V}$ *is a vector potential for* $\mathbf{V}$.

3) Let the curve $\mathcal{K}$ *be the boundary of the square of vertices* $(0,0,0)$*,* $(0,\pi,0)$*,* $(0,\pi,\pi)$ *and* $(0,0,\pi)$*, in the given succession. Find the circulation*

$$\oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, ds.$$

4) Let

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 \mid 1 \le x^2+y^2+z^2 \le 4\right\}.$$

Find the flux of the vector field

$$\mathbf{U}(x,y,z) = (x,y,z) + \mathbf{V}(x,y,z), \qquad (x,y,z) \in \mathbb{R}^3,$$

through $\partial\Omega$*, when the unit normal vector* $\mathbf{n}$ *of* $\partial\Omega$ *is pointing away from* Ω.

A Divergence, rotation, vector potential, circulation, flux.

D Apply Stokes's theorem and Gauß's theorem.

I 1) It follows immediately that

$$\text{div}\, \mathbf{V} = \bigtriangledown \cdot \mathbf{V} = 0.$$

Then

$$\begin{aligned}
\textbf{rot }\mathbf{V} &= \bigtriangledown \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \cos y - \sin z & \cos z - \sin x & \cos x - \sin y \end{vmatrix} \\
&= (-\cos y + \sin z, -\cos z + \sin x, -\cos x + \sin y) \\
&= -(\cos y - \sin z, \cos z - \sin x, \cos x - \sin y) = -\mathbf{V}.
\end{aligned}$$

2) If we choose $\alpha = -1$ in 1), then

$$\bigtriangledown \times (-\mathbf{V}) = \mathbf{V},$$

and it follows that $-\mathbf{V}$ is a vector potential for $\mathbf{V}$.

3) Here we give two variants.

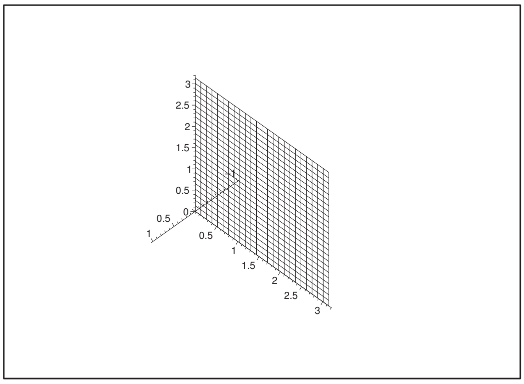


Figure 6: The square $\mathcal{K}$.

a) *Stokes's theorem.* The square $\mathcal{K}$ lies in the YZ-plane, and the unit normal vector is in the chosen orientation given by

$$\mathbf{n} = (1,0,0).$$

Then by 1) and Stokes's theorem,

$$\begin{aligned}\oint_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,ds &= \int_{\tilde{\mathcal{K}}} \mathbf{n}\cdot\mathbf{rot}\;\mathbf{V}\,dS = \int_{\tilde{\mathcal{K}}} (-\cos y + \sin z)\,dS \\ &= \int_0^{\pi}\left\{\int_0^{\pi}(-\cos y)\,dy\right\}dz + \int_0^{\pi} dy\cdot\int_0^{\pi}\sin z\,dz \\ &= 0+\pi\cdot[-\cos z]_0^{\pi} = 2\pi.\end{aligned}$$

b) *Straight forward computation of the line integral.* The curve $\mathcal{K}$ is composed of the curves

$$\begin{aligned}&\mathcal{K}_1: \quad \mathbf{r}_1(t) = (0,t,0), && t\in[0,\pi], \quad \mathbf{t} = (0,1,0),\\ &\mathcal{K}_2: \quad \mathbf{r}_2(t) = (0,\pi,t), && t\in[0,\pi], \quad \mathbf{t} = (0,0,1),\\ &\mathcal{K}_3: \quad \mathbf{r}_3(t) = (0,\pi-t,\pi), && t\in[0,\pi], \quad \mathbf{t} = (0,-1,0),\\ &\mathcal{K}_4: \quad \mathbf{r}_4 = (0,0,\pi-t), && t\in[0,\pi], \quad \mathbf{t} = (0,0,-1).\end{aligned}$$

Then by insertion,

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,ds &= \int_{\mathcal{K}_1}(\cos z - \sin x)ds + \int_{\mathcal{K}_2}(\cos x - \sin y)ds \\ &\qquad + \int_{\mathcal{K}_3}(-\cos z + \sin x)ds + \int_{\mathcal{K}_4}(-\cos x + \sin y)ds \\ &= \int_0^{\pi}(1-0)dt + \int_0^{\pi}(1-\sin\pi)dt \\ &\qquad + \int_0^{\pi}(-\cos\pi + \sin 0)dt + \int_0^{\pi}(-1+\sin 0)dt \\ &= \pi+\pi+\pi-\pi = 2\pi.\end{aligned}$$

c) It follows from 1) and Gauß's theorem that

$$\begin{aligned}\text{flux} &= \int_{\partial\Omega} \mathbf{U}\cdot\mathbf{n}\,dS = \int_\Omega \text{div}\,\mathbf{U}\,d\Omega = \int_\Omega \{3+\ \text{div}\,\mathbf{V}\}\,d\Omega \\ &= \int_\Omega (3+0)\,d\Omega = 3\,\text{vol}(\Omega) = 3\cdot\frac{4\pi}{3}\cdot(2^3-1^3) = 28\pi.\end{aligned}$$

Example 2.12 **1.** *Find the rotation of the vector field*

$$\mathbf{U}(x,y,z) = (-yz, 0, xy), \qquad (x,y,z)\in\mathbb{R}^3,$$

and show that $\mathbf{U}$ *is not a gradient field.*
A space curve $\mathcal{K}$ *is given by the parametric description*

$$(x,y,z) = \mathbf{r}(t) = \left(\cos^3 t, 3\cos t, \sin^3 t\right), \qquad t\in\left[0,\frac{\pi}{2}\right].$$

2. *Compute the tangential line integral*

$$\int_{\mathcal{K}} \mathbf{U}\cdot d\mathbf{x}.$$

3. *Find a function* $G(x,z)$*,* $(x,z)\in\mathbb{R}^2$*, such that the vector field*

$$\mathbf{W}(x,y,z) = (0, y\,G(x,z), 0), \qquad (x,y,z)\in\mathbb{R}^3,$$

is a vector potential for $\mathbf{U}$*.*

A Rotation; tangential line integral; vector potential.

D Use the standard methods in the former two questions and check the conditions of a vector potential in the latter question.

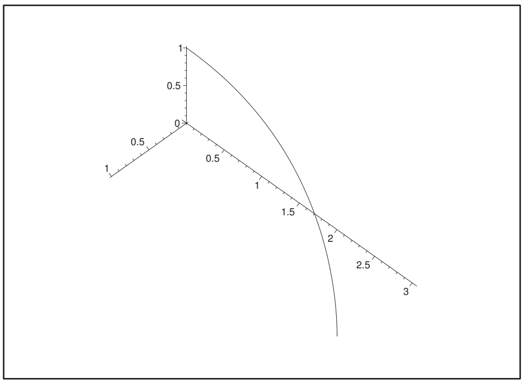


Figure 7: The space curve $\mathcal{K}$.

I 1) It follows immediately that $\mathbf{U}$ is divergence free. Then

$$\mathbf{rot}\ \mathbf{U} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -yz & 0 & xy \end{vmatrix} = (x, -y-y, z) = (x, -2y, z).$$

It follows from $\mathbf{rot}\ \mathbf{U}(\mathbf{x}) \neq \mathbf{0}$ for $\mathbf{x} \neq \mathbf{0}$, that $\mathbf{U}$ is not a gradient field.

2) We get from

$$\mathbf{r}(t) = \left(\cos^3 t, 3\cos t, \sin^3 t\right)$$

that

$$\mathbf{r}'(t) = \left(-3\cos^2 t \sin t, -3\sin t, 3\sin^2 t \cos t\right),$$

and the tangential line integral is reduced to

$$\begin{aligned}
\int_{\mathcal{K}} \mathbf{U}\cdot d\mathbf{x} &= 3\cdot 3\int_0^{\frac{\pi}{2}} (-\cos t\sin^3 t, 0, \cos^4 t)\cdot(-\cos^2 t\sin t, -\sin t, \sin^2 t\cos t)\,dt \\
&= 9\int_0^{\frac{\pi}{2}} \left\{\cos^3 t\sin^4 t+\cos^5 t\sin^2 t\right\}\,dt \\
&= 9\int_0^{\frac{\pi}{2}} \cos^3 t\sin^2\{\sin^2 t+\cos^2 t\}\,dt = 9\int_0^{\frac{\pi}{2}} \cos^3 t\sin^2 t\,dt \\
&= 9\int_0^{\frac{\pi}{2}} (1-\sin^2 t)\sin^2 t\cdot\cos t\,dt = 9\int_0^1 (u^2-u^4)\,du \\
&= 9\left(\frac{1}{3}-\frac{1}{5}\right) = 9\cdot\frac{2}{15}=\frac{6}{5}.
\end{aligned}$$

3) If $\mathbf{W}(x,y,z)=(0,y\,G(x,z),0)$, then

$$\mathbf{rot}\ \mathbf{W} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & y\,G(x,z) & 0 \end{vmatrix} = (-y\,G_z'(x,z), 0, y\,G_x'(x,z))$$

is equal to $\mathbf{U}$ for

$$G_z'(x,z)=z \qquad \text{and} \qquad G_x'(x,z)=x,$$

hence by integration

$$G(x,z)=\frac{1}{2}\,z^2+\varphi_1(x)=\frac{1}{2}\,x^2+\varphi_2(z),$$

and by a rearrangement

$$\frac{1}{2}\,z^2-\varphi_2(z)=\frac{1}{2}\,x^2-\varphi_1(x)=\ \text{constant},$$

so

$$G(x,z)=\frac{1}{2}\,x^2+\frac{1}{2}\,z^2+C, \qquad C \text{ arbitrary constant.}$$

REMARK. If we instead apply then we should first notice that $\mathbf{U}$ is divergence free in the star shaped (convex) domain $\mathbb{R}^3$ containing $\mathbf{0}$. This implies the existence of the vector potentials and that one of these can be found by the formula

$$\mathbf{W}_0(\mathbf{x})=\int_0^1 \mathbf{T}(\tau\,\mathbf{x})\,d\tau, \qquad \text{where } \mathbf{T}(\mathbf{x})=\mathbf{U}(\mathbf{x})\times\mathbf{x}.$$

First calculate

$$\mathbf{T}(\mathbf{x})=\mathbf{U}(\mathbf{x})\times\mathbf{x}=\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -yz & 0 & xy \\ x & y & z \end{vmatrix} = \left(-xy^2, x^2y+yz^2, -y^2z\right).$$

All coordinates are precisely of degree 3, thus by an integration with respect to τ,

$$\mathbf{W}_0(\mathbf{x}) = \int_0^1 \mathbf{T}(\tau\,\mathbf{x})\,d\tau = \mathbf{T}(\mathbf{x})\int_0^1 \tau^3\,d\tau = \left(-\frac{1}{4}\,xy^2, \frac{1}{4}\,(x^2+z^2)y, -\frac{1}{4}\,y^2z\right).$$

We see that $\mathbf{W}_0(\mathbf{x})$ is a vector potential for $\mathbf{U}(\mathbf{x})$. It is, however, not of the wanted type. $\Diamond$

Example 2.13 *Two vector fields* $\mathbf{V}$, $\mathbf{W} : \mathbb{R}^3 \to \mathbb{R}^3$ *are given by*

$$\mathbf{V}(x,y,z) = \left(e^y \sin z, x\,e^y \sin z, x\,e^y \cos z\right),$$

$$\mathbf{W}(x,y,z) = \left(x + 2x\,e^y \cos z, -2e^y \cos z, -z + z^3\right).$$

1) Find the divergence and the rotation of both vector fields.

2) Show that $\mathbf{V}$ *is a gradient field and find all its integrals.*

3) Compute the tangential line integral

$$\int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,ds,$$

where $\mathcal{K}$ *is the broken line composed of the three line segments: from* $(0,0,0)$ *to* $(1,0,0)$*, from* $(1,0,0)$ *to* $(1,2,0)$*, and from* $(1,2,0)$ *to* $\left(1,2,\frac{\pi}{2}\right)$.

4) Show the existence of a constant α*, such that* $\alpha\mathbf{W}$ *is a vector potential for* $\mathbf{V}$*; find* α.

5) Let $\mathcal{F}$ *be the sphere of centrum* $(0,0,0)$ *and radius* 3*. Find the flux*

$$\int_{\mathcal{F}} \mathbf{W}\cdot\mathbf{n}\,dS,$$

where the unit normal vector $\mathbf{n}$ *is pointing away from the centrum of* $\mathcal{F}$.

A Divergence and rotation; gradient field and integrals; tangential line integral; vector potential; flux.

D It is in some sense better to go through the example in an other succession than the above. If we let 4) follow immediately after 1), then it becomes obvious. We give three variants of 2) and two variants of 3), while 5) is given in 3 variants.

I 1. We get by straightforward computations,

$$\begin{aligned}
\text{div}\,\mathbf{V} &= 0 + x\,e^y \sin z - x\,e^y \sin z = 0,\\
\mathbf{rot}\ \mathbf{V} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^y \sin z & xe^y \sin z & xe^y \cos z \end{vmatrix} = \begin{pmatrix} xe^y \cos z - xe^y \ \cos z \\ e^y \cos z - e^y \cos z \\ e^y \sin z - e^y \sin z \end{pmatrix}\\
&= \mathbf{0},\\
\text{div}\,\mathbf{W} &= 1 + 2e^y \cos z - 2e^y \cos z - 1 + 3z^2 = 3z^2,\\
\mathbf{rot}\ \mathbf{W} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + 2xe^y \cos z & -2e^y \cos z & -z + z^3 \end{vmatrix}\\
&= (0 - 2e^y \sin z, -2xe^y \sin z - 0, 0 - 2xe^y \cos z)\\
&= -2\,(e^y \sin z, xe^y \sin z, xe^y \cos z) = -2\mathbf{V}.
\end{aligned}$$

4. As $\mathbf{rot}\ \mathbf{W} = \bigtriangledown \times \mathbf{W} = -2\mathbf{V}$, we have $\bigtriangledown \times \left(-\frac{1}{2}\,\mathbf{W}\right) = \mathbf{V}$, and it follows immediately that $-\frac{1}{2}\,\mathbf{W}$ is a vector potential for $\mathbf{V}$, and that $\alpha = -\frac{1}{2}$.

2. As $\mathbf{rot}\ \mathbf{V} = \mathbf{0}$ and $\mathbb{R}^3$ is star shaped (it is even convex), $\mathbf{V}$ is a gradient field. Its integrals may be found in one of the following three ways:

First variant. *Indefinite integration.*

$$F_1(x,y,z) = \int V_x(x,y,z)\,dx = \int e^y \ \sin z\,dx = x\,e^y \sin z,$$

where

$$\bigtriangledown F_1 = (e^y \sin z, x\,e^y \sin z, x\,e^y \cos z) = \mathbf{V},$$

proving that F_1 is an integral of $\mathbf{V}$ and all integrals are given by

$$F(x,y,z) = x\,e^y \sin z + C, \qquad C \text{ arbitrary constant.}$$

Second variant. *Manipulation, using the rules of calculations.* It follows immediately from

$$\begin{aligned}
\mathbf{V}\cdot d\mathbf{x} &= e^y \sin z\,dx + x^y \sin z\,dy + xe^y \cos z\,dz\\
&= e^y \cdot \sin z\,dx + x \cdot \sin z \cdot d\,(e^y) + xe^y\,d(\sin z)\\
&= d\,(xe^y \sin z) = d\,(xe^y \sin z + C)\,,
\end{aligned}$$

that the integrals are given by

$$F(x,y,z) = x\,e^y \sin z + C, \qquad C \text{ arbitrary constant.}$$

Third variant. *Integration along a broken line*

$$(0,0,0) \longrightarrow (x,0,0) \longrightarrow (x,y,0) \longrightarrow (x,y,z).$$

This gives the candidates

$$\begin{aligned} F(x,y,z) &= C + \int_0^x 0\,dt + \int_0^y 0\,dt + \int_0^z x\,e^y \cos t\,dt \\ &= x\,e^y \sin z + C. \end{aligned}$$

Now, we have proved above that the integrals exist, so we conclude that these are the set of all integrals when the arbitrary constant $C \in \mathbb{R}$ varies.

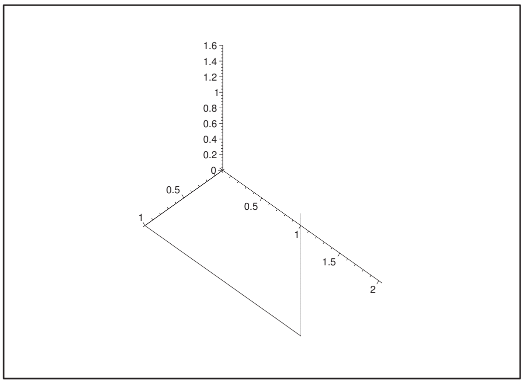


Figure 8: The curve $\mathcal{K}$.

3. We get by the *gradient theorem* that

$$\int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,ds = F\left(1,2,\frac{\pi}{2}\right) - F(0,0,0) = 1\cdot e^2\cdot\sin\frac{\pi}{2} - 0 = e^2.$$

ALTERNATIVELY, write $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3$, where

$$\mathcal{K}_1: \quad (x,y,z) = (t,0,0), \quad t\in[0,1], \qquad \mathbf{t} = (1,0,0),$$

$$\mathcal{K}_2: \quad (x,y,z) = (1,t,0), \quad t\in[0,2], \qquad \mathbf{t} = (0,1,0),$$

$$\mathcal{K}_3: \quad (x,y,z) = (1,2,t), \quad t\in\left[0,\frac{\pi}{2}\right], \quad \mathbf{t} = (0,0,1),$$

thus

$$\int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,ds = \int_0^1 0\,dt + \int_0^2 0\,dt + \int_0^{\frac{\pi}{2}} 1\cdot e^2\cos t\,dt = e^2.$$

4. This was answered previously.

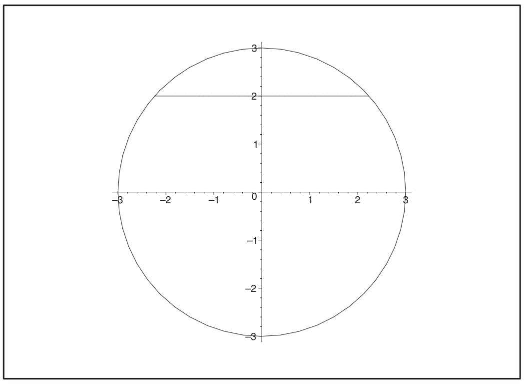


Figure 9: The intersection of the ball Ω with the (X,Z)-plane.

5. The sphere $\mathcal{F}$ encloses the ball Ω, so it follows from Gauß's theorem and 1) that

$$\int_{\mathcal{F}} \mathbf{W}\cdot\mathbf{n}\,dS = \int_{\Omega} \text{div}\;\mathbf{W}\,d\Omega = \int_{\Omega} 3z^2\,d\Omega.$$

This space integral is then computed in three different ways:

First variant. *the slicing method.* At height $z\in[-3,3]$ the ball is intersected into a disc $B(z)$ of radius $\varrho=\sqrt{9-z^2}$, thus

$$\begin{aligned}
\text{flux} &= \int_{\Omega} 3z^2\,d\Omega = \int_{-3}^{3} 3z^2\left\{\int_{B(z)} dS\right\}dz = \int_{-3}^{3} 3z^2\,\text{areal}\;B(z)\,dz\\
&= \int_{-3}^{3} 3z^2\cdot\pi\left(9-z^2\right)\,dz = 3\pi\int_{-3}^{3}\left(9z^2-z^4\right)\,dz = 3\pi\left[3z^2-\frac{z^5}{5}\right]_{-3}^{3}\\
&= 3\pi\cdot 2\left(81-\frac{3\cdot 81}{5}\right) = 6\pi\cdot 81\cdot\left(1-\frac{3}{5}\right) = \frac{972\pi}{5}.
\end{aligned}$$

Second variant. *The post method.* We get in polar coordinates

$$\begin{aligned}
\text{flux} &= \int_{\Omega} 3z^2\,dz = \int_{0}^{2\pi}\left\{\int_{0}^{3}\left(\int_{-\sqrt{9-\varrho^2}}^{\sqrt{9-\varrho^2}} 3z^2\,dz\right)\varrho\,d\varrho\right\}d\varphi\\
&= 2\pi\int_{0}^{3}\left[z^3\right]_{-\sqrt{9-\varrho^2}}^{\sqrt{9-\varrho^2}}\varrho\,d\varrho = 2\pi\int_{0}^{3}\left(9-\varrho^2\right)^{\frac{3}{2}}\cdot 2\varrho\,d\varrho\\
&= 2\pi\left[-\frac{2}{5}\left(9-\varrho^2\right)^{\frac{5}{2}}\right]_{0}^{3} = \frac{4}{5}\,\pi\cdot 9^{\frac{5}{2}} = \frac{4\pi}{5}\cdot 3^5 = \frac{972\pi}{5}.
\end{aligned}$$

Third variant. *Spherical coordinates.* When we use these we get

$$\begin{aligned}
\text{flux} &= \int_{\Omega} 3z^2\,dz = \int_{0}^{2\pi}\left\{\int_{0}^{\pi}\left(\int_{0}^{3} 3r^2\cos^2\theta\cdot r^2\sin\theta\,dr\right)d\theta\right\}d\varphi\\
&= 2\pi\int_{0}^{\pi} 3\cos^2\theta\sin\theta\,d\theta\cdot\int_{0}^{3} r^5\,dr = 2\pi\left[-\cos^3\theta\right]_{0}^{\pi}\cdot\left[\frac{r^5}{5}\right]_{0}^{3}\\
&= 2\pi\cdot 2\cdot\frac{3^5}{5} = \frac{972\pi}{5}.
\end{aligned}$$

Remark. A direct computation of the flux by the definition alone looks impossible, because

$$\begin{aligned}
\mathbf{W}\cdot\mathbf{n} &= \left(x+2xe^y\cos z, -2e^y\cos z, -z+z^3\right)\cdot\frac{1}{3}\,(x,y,z)\\
&= \frac{1}{3}\left\{x^2+2x^2e^y\cos z-2ye^y\cos z-z^2+z^4\right\},
\end{aligned}$$

and what then? $\diamond$

Example 2.14 *Given the vector field*

$$\mathbf{V}(x,y,z) = \begin{cases} (-y,x,0), & x^2+y^2<a^2, \\[2mm] \dfrac{a^2}{x^2+y^2}\,(-y,x,0), & x^2+y^2 \geq a^2. \end{cases}$$

1) Let $\mathcal{K}$ be the circle of constant values of the coordinates ϱ and z, and with a positive orientation with respect to the unit vector $\mathbf{e}_z$. Prove that

$$\oint_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,ds = \begin{cases} 2\pi\varrho^2, & \varrho < a, \\[2mm] 2\pi a^2, & \varrho \geq a. \end{cases}$$

2) Show that $(0,0,W)$, where

$$W(x,y,z) = \begin{cases} \dfrac{1}{2}\,(a^2-x^2-y^2), & x^2+y^2<a^2, \\[2mm] a^2\,\ln\dfrac{a}{\sqrt{x^2+y^2}}, & x^2+y^2 \geq a^2, \end{cases}$$

is a vector potential for $\mathbf{V}$.

A Circulation along a curve; vector potential.

D Compute the circulation straight forward. (Consider if it is possible to use Stokes's theorem instead. Show that $\bigtriangledown \times (W\,\mathbf{e}_z) = \mathbf{V}$.

I 1) Since $\mathbf{V}$ does not depend on z, we may assume that $\mathcal{K}$ lies in the XY-plane. Then by Stokes'e theorem,

$$\oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, ds = \int_B \mathbf{e}_z \cdot \mathbf{rot}\ \mathbf{V}\, dS,$$

where B denoted the disc of radius ϱ. If $\varrho > a$, the right hand side does not look nice, so we compute instead the circulation by the definition. Let

$$\mathcal{K}: \quad (\varrho\,\cos\varphi, \varrho\,\sin\varphi), \qquad \varphi \in [0, 2\pi].$$

Then

$$\mathbf{t}\, ds = \varrho\,(-\sin\varphi, \cos\varphi)\, d\varphi.$$

Then for $\varrho < a$,

$$\begin{aligned}\oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, ds &= \int_0^{2\pi} \varrho\,(-\sin\varphi, \cos\varphi) \cdot \varphi\,(-\sin\varphi, \cos\varphi)\, d\varphi \\ &= \int_0^{2\pi} \varrho^2 \left(\sin^2\varphi + \cos^2\varphi\right)\, d\varphi \\ &= \varrho^2 \int_0^{2\pi} d\varphi = 2\pi\varrho^2 \qquad \text{for } \varrho < a,\end{aligned}$$

and we have for $\varrho \geq a$

$$\begin{aligned}\oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, ds &= \int_0^{2\pi} \frac{a^2}{\varrho^2} \cdot \varrho\,(-\sin\varphi, \cos\varphi) \cdot \varrho\,(-\sin\varphi, \cos\varphi)\, d\varphi \\ &= a^2 \int_0^{2\pi} d\varphi = 2\pi a^2, \qquad \text{for } \varrho \geq a,\end{aligned}$$

thus the first claim has been proved.

2) If $x^2 + y^2 < a^2$, then

$$\bigtriangledown \times (W\,\mathbf{e}_z) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & \frac{1}{2}\left(a^2 - (x^2 + y^2)\right) \end{vmatrix} = (-y, x, 0) = \mathbf{V},$$

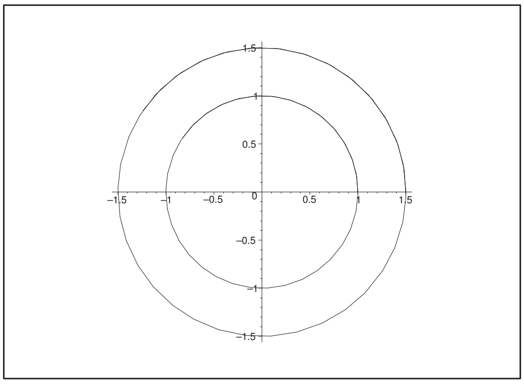


Figure 10: The curve $\mathcal{K}$ in the XY-plane for $a = 1$ and $\varrho = 1.5$.

and if $x^2 + y^2 > a^2$, then

$$\begin{aligned} \nabla \times (W\,\mathbf{e}_z) &= \left| \begin{array}{ccc} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & a^2 \ln\left(\dfrac{a}{\sqrt{x^2+y^2}}\right) \end{array} \right| \\ &= \frac{a^2}{2}\left(-\frac{\partial}{\partial t}\left(x^2+y^2\right), \, \frac{\partial}{\partial x}\left(x^2+y^2\right), \, 0\right) = \frac{a^2}{x^2+y^2}\,(-y,x,0) = \mathbf{V}. \end{aligned}$$

By the continuity from the inside and from the outside we get

$$\nabla \times (W\,\mathbf{e}_z) = (-y,x,0) = \mathbf{V} \qquad \text{for } x^2+y^2 = a^2.$$

Hence we have proved that $W\,\mathbf{e}_z$ is a vector potential for $\mathbf{V}$.

3 Green's identities

Example 3.1 *Consider a bounded domain $\Omega \subset \mathbb{R}^3$ with its boundary consisting of $m+1$ disjoint surfaces $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_m$, such that $\mathcal{F}_0$ surrounds all the others.*
We shall find a function w, which in Ω^0 fulfils Poisson's equation

$$\nabla^2 w = p,$$

and which has constant values on the surfaces $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_m$. Let Φ_i denote the flux of ∇w through $\mathcal{F}_i$, i.e.

$$\Phi_i = \int_{\mathcal{F}_i} \frac{\partial w}{\partial n}\, dS.$$

1. *Let the function p be given, and assume that w is zero on $\mathcal{F}_0$, and for each $i \in \{1, \dots, m\}$ either Φ_i or the value of w is given.*

show that w is uniquely determined.

Then let Ω be an unbounded domain with its boundary consisting of m disjoint and bounded surfaces $\mathcal{F}_1, \dots, \mathcal{F}_m$. Then the uniqueness theorem proved above also holds when the condition on $\mathcal{F}_0$ is replaced by the following:
There exist positive constants C_1, C_2, such that

$$\|\mathbf{x}\|\,|w(\mathbf{x})| \le C_1 \quad \textit{and} \quad \|\mathbf{x}\|^2 \| \nabla w(\mathbf{x})\| \le C_2 \qquad \textit{for all } \mathbf{x} \in \Omega.$$

2. *Prove this by considering $\Omega(R) = \Omega \cap \overline{K}(\mathbf{0}; R)$ and then let R tend to plus infinity.*

A Uniqueness theorem for a mixed Dirichlet/Neumann problem for Poisson's equation.

D Assume that w and $\tilde{w}$ are solutions. Put $f = v - \tilde{w}$ and apply Green's first theorem by choosing $g = f$ and applying that $f(\mathbf{x}) = 0$ on every $\mathcal{F}_i$.

In 2) we estimate the integrand in $\int_{\partial\Omega(R)} f \dfrac{\partial f}{\partial n}\, dS$.

Remark 1. The example is dealing with a *uniqueness theorem* within a smaller class of functions than the mathematically most natural class. Therefore, on cannot expect that there actually exists a solution within this class. The problem is that the Neumann problem in some cases is difficult to treat. However, we can succeed if we have a boundary surface $\mathcal{F}_i$ with a Dirichlet condition instead, i.e. $f(\mathbf{x}) = \alpha_i$ on $\mathcal{F}_i$. The situation is worse if we are given the flux Φ_i on $\mathcal{F}_i$, because then we cannot in general conclude that $f(\mathbf{x})$ is equal to some (unknown) constant on $\mathcal{F}_i$. This is in general not the case, so we shall usually only expect to be able to show the uniqueness and not the existence of a solution within the given class of functions. $\Diamond$

I 1) First give the problem a mathematical description:

$$(4)\quad \left\{ \begin{array}{lll} \nabla^2 w = p, & \text{in } \Omega^0 & \text{Poisson equation} \\ \left\{ \begin{array}{ll} w(\mathbf{x}) = 0, \ \text{in } \mathcal{F}_0 & i = 0 \\ w(\mathbf{x}) = \alpha_i, \ \text{in } \mathcal{F}_i & i \in \{i_1, \dots, i_k\} \end{array} \right\} & & \text{Dirichlet conditions} \\ \left\{ \begin{array}{ll} \int_{\mathcal{F}_i} \dfrac{\partial w}{\partial n}\, dS = \Phi_i, & i \notin \{i_1, \dots, i_k\} \\ w(\mathbf{x}) = \alpha_i \text{ in } \mathcal{F}_i & i \notin \{i_1, \dots, i_k\} \end{array} \right. & & \begin{array}{l} \text{Neumann conditions} \\ \text{additional condition.} \end{array} \end{array} \right.$$

It will be convenient to put

$$A = \bigcup_{j=1}^{k} \mathcal{F}_{i_j} \cup \mathcal{F}_0 \quad \text{og} \quad B = \partial\Omega \setminus A, \quad \text{dvs. } \partial\Omega = A \cup B, \text{ disjunkt.}$$

Assume that w and $\tilde{w}$ are solutions of (4). By putting $f = w - \tilde{w}$ it follows by the linearity and the additional condition that f satisfies

$$(5) \quad \begin{cases} \nabla^2 f = 0, & \\ f(\mathbf{x}) = 0 & \text{på } A \cup B = \partial\Omega, \\ \int_{\mathcal{F}_i} \dfrac{\partial f}{\partial n}\, dS = 0 & i \notin \{i_1, \dots, i_k\}. \end{cases}$$

Choose $g = f$ in Green's first formula. Then

$$\int_\Omega \{f \nabla^2 f + \nabla f \cdot \nabla f\}\, d\Omega = \int_\Omega \{0 + \| \nabla f\|^2\}\, d\Omega = \int_{\partial\Omega} f \frac{\partial f}{\partial n}\, dS = 0,$$

because $f(\mathbf{x}) = 0$ on $\partial\Omega$, thus

$$\int_\Omega \| \bigtriangledown f\|^2 d\Omega = 0.$$

Since $\| \bigtriangledown f\|^2$ is continuous and nonnegative, we must have $\bigtriangledown f = \mathbf{0}$, and we conclude that f is a constant. Now, f is continuous and zero on the boundary, so f must be identical zero, thus $w = \tilde{w}$, and we have proved the uniqueness in the bounded case.

Remark 2. Note that the flux Φ_i through some of the surfaces $\mathcal{F}_i$ does not enter the argument at all, since we are only using the strong additional condition that $w(\mathbf{x}) = \alpha_i$ (the same though unknown constant) for $i \notin \{i_1, \dots, i_k\}$. Hence the problem is formally over-determined, since we do not apply all our information. (If this information is not in agreement with that we only get the zero solution, we clearly have a problem. This illustrates what is mathematically "wrong" with this example). $\Diamond$

2) Then consider the unbounded case with the additional growth conditions as a replacement of the missing surface $\mathcal{F}_0$.

When we take the intersection of Ω with the ball $\overline{K}(\mathbf{0}; R)$, we get a bounded domain $\Omega(R)$. It is left to the reader to sketch the situation on a figure.

Then split the boundary of $\Omega(R)$ in the following way

$$A(R) = \overline{K}(\mathbf{0}; R) \cap \bigcup_{j=1}^k \mathcal{F}_{i_j},$$

$$B(R) = \overline{K}(\mathbf{0}; R) \cap \bigcup_{i \notin \{i_1,\dots,i_k\}} \mathcal{F}_i,$$

$$C(R) = \partial\overline{K}(\mathbf{0}; R) \setminus \{A(R) \cup B(R)\},$$

where we have Dirichlet conditions on $\mathcal{F}_i$ for $i \in \{i_1, \dots, i_k\}$ and Neumann conditions on $\mathcal{F}_i$ for $i \notin \{i_1, \dots, i_k\}$. Apart from the fact that we do not know the behaviour on $C(R)$, the problem can with some modifications be written as in (4).

Let w and $\tilde{w}$ be solutions. We put again $f = w - \tilde{w}$. Then f satisfies (5) with the modifications that $A \cup B = \partial\Omega$ is replaced by $A(R) \cup B(R)$ $[\subseteq \partial\Omega_R]$, and $\mathcal{F}_i$ is replaced by $\mathcal{F}_i \cap \overline{K}(\mathbf{0}; R)$.

Choose as before $g = f$ in Green's first formula. Then

$$\int_{\Omega(R)} \| \bigtriangledown \|^2 d\Omega = \int_{\partial\Omega(R)} f\,\frac{\partial f}{\partial n}\,dS = \int_{A(R)} f\,\frac{\partial f}{\partial n}\,dS + \int_{B(R)} f\,\frac{\partial f}{\partial b}\,dS + \int_{C(R)} f\,\frac{\partial f}{\partial n}\,dS.$$

Since f is zero on $A(R)$ and $B(R)$, this is reduced to

$$\int_{\Omega(R)} \| \bigtriangledown f\|^2 d\Omega = \int_{C(R)} f\,\frac{\partial f}{\partial n}\,dS,$$

which is not necessarily zero.

We notice that according to the additional conditions we have for $\mathbf{x} \in C(R)$ that

$$|f(\mathbf{x})| = |w(\mathbf{x}) - \tilde{w}(\mathbf{x})| \le |w(\mathbf{x})| + \|\tilde{w}(\mathbf{x})| \le \frac{2C_1}{\|\mathbf{x}\|} = \frac{2C_1}{R},$$

and

$$\left|\frac{\partial f}{\partial n}\right| \le \| \bigtriangledown f\| = \| \bigtriangledown w - \bigtriangledown \tilde{w}\| \le \| \bigtriangledown w\| + \| \bigtriangledown \tilde{w}\| \le \frac{2C_2}{R^2},$$

and we obtain the estimates

$$\begin{aligned}\left|\int_{C(R)} f\,\frac{\partial f}{\partial n}\,dS\right| &\le \int_{C(R)} |f|\cdot\left|\frac{\partial f}{\partial n}\right| dS \le \int_{C(R)} \frac{2C_1}{R}\cdot\frac{2C_2}{R^2}\,dS \\ &= \frac{4C_1C_2}{R^3}\,\text{areal}(C(R)) \le \frac{4C_1C_2}{R^3}\,\text{area}(\partial\overline{K}(\mathbf{0};R)) \\ &= \frac{4V_1C_2}{R^3}\cdot 4\pi R^2 = \frac{16\pi C_1C_2}{R} \to 0 \qquad \text{for } R\to+\infty,\end{aligned}$$

from which we conclude that

$$\int_\Omega \| \bigtriangledown f\|^2 d\Omega = \lim_{R\to\infty}\int_{\Omega(R)} \| \bigtriangledown f\|^2 d\Omega = \lim_{R\to+\infty}\int_{C(R)} f\,\frac{\partial f}{\partial n}\,dS = 0.$$

Notice that as $\| \bigtriangledown f\|^2 \ge 0$, we can take this limit to find the value of the improper integral

$$\int_\Omega \| \bigtriangledown f\|^2 d\Omega = 0.$$

Since $\| \bigtriangledown f\|^2 \ge 0$ is continuous we conclude as above that $\bigtriangledown f = 0$, i.e. f is a constant. Finally, it follows from the boundary value that $f(\mathbf{x}) = 0$ for $\mathbf{x}\in\Omega$, hence $w(\mathbf{x}) = \tilde{w}(\mathbf{x})$ in Ω, and we have proved the uniqueness.

Remark 3. As mentioned above this is not a proof of the existence. Consider as an extreme example the problem

$$\begin{cases} \bigtriangledown^2 w = p, & \text{Poisson equation} \\ \int_{\mathcal{F}} \dfrac{\partial w}{\partial n}\,dS = \Phi & \text{Neumann problem on } \mathcal{F} \\ \|\mathbf{x}\|\cdot|w(\mathbf{x})| \le C_1 & \text{for } \mathbf{x}\in\Omega, \\ \|\mathbf{x}\|^{2+\varepsilon}\| \bigtriangledown w(\mathbf{x})\| \le C_2 & \text{for } \mathbf{x}\in\Omega \end{cases}$$

Apart from the fact that the exponent 2 has been changed to $2+\varepsilon$ of convergency reasons, this is a special case of 2) above.

When we integrate $\Omega(R)$ and choose $g=1$ in Green's formula, we get that

$$\int_{\Omega(R)} \{1\cdot\bigtriangledown^2 w + \bigtriangledown 1\cdot\bigtriangledown f\}\,d\Omega = \int_{\partial\Omega(R)} \frac{\partial w}{\partial n}\,dS,$$

which is reduced to

$$\int_{\Omega(R)} p\,d\Omega = \int_{C(R)} \frac{\partial w}{\partial n}\,dS + \int_{\mathcal{F}\cap\Omega(R)} \frac{\partial w}{\partial n}\,dS.$$

The former term on the right hand side is estimated by

$$\left|\int_{C(R)} \frac{\partial w}{\partial n}\, dS\right| \le \int_{C(R)} \| \bigtriangledown w\|\, dS \le \frac{C_2}{R^{2+\varepsilon}} \cdot 4\pi R^2 = \frac{4\pi C_2}{R^{\varepsilon}} \to 0 \qquad \text{for } R \to +\infty,$$

and the latter term clearly converges towards

$$\lim_{R\to+\infty} \int_{\mathcal{F}\cap\Omega(R)} \frac{\partial w}{\partial n}\, dS = \int_{\mathcal{F}} \frac{\partial w}{\partial n}\, dS = \Phi,$$

and we get the compatibility condition

$$\int_{\Omega} p\, d\Omega = \Phi,$$

proving that p and Φ are not independent of each other.

Notice that if we also have a Dirichlet condition and the improper integral $\int_\Omega p\, d\Omega$ is convergent, then the unknown flux through the Dirichlet boundary forces that the compatibility condition is fulfilled. $\Diamond$

Remark 4. The example has been formulated from a physical point of view. In general, the corresponding mathematical problem in the bounded case is described as follows:

$$\begin{cases} \bigtriangledown^2 w = p, & \text{in } \Omega^0, & \text{Poisson,} \\ w(\mathbf{x}) = 0, & \text{in } \mathcal{F}_0, & \text{Dirichlet,} \\ w(\mathbf{x}) = \alpha_i, & \text{in } \mathcal{F}_i \text{ for } i \in \{i_1, \dots, i_k\}, & \text{Dirichlet,} \\ \dfrac{\partial w}{\partial n} = h_i(\mathbf{x}), & \text{in } \mathcal{F}_i \text{ for } i \notin \{i_1, \dots, i_k\}, & \text{Neumann,} \end{cases}$$

or similarly $\left(\text{e.g. } \dfrac{\partial w}{\partial n} = h_0(\mathbf{x}) \text{ in } \mathcal{F}_0\right)$.

If the boundary conditions are only of Neumann type, we must add a compatibility condition:

$$\int_{\Omega} p\, d\Omega = \int_{\partial\Omega} h\, dS,$$

where $h(\mathbf{x}) = h_i(\mathbf{x})$ på $\mathcal{F}_i$.

Notice that we do *not* assume that w is constant on the Neumann boundaries.

Assuming that p and h are nice functions we can prove that we have both an existence and a uniqueness theorem for the problem. For the pure Dirichlet problem the proof is classical known. However, if just one Neumann boundary occurs, the proof becomes very difficult. One shall e.g. apply *Hopf's maximum principle:* In a connected domain Ω a non-constant harmonic function w only attains its maximum values (if they exist) on the boundary $\partial\Omega$, and we have at such a maximum point

$$\frac{\partial w}{\partial n} > 0. \qquad \Diamond$$

Example 3.2 *Let Ω be a domain in the space for a given non-constant function*

$$g : \Omega \to [0, +\infty[.$$

We shall find a function w, which satisfies

(6) $\bigtriangledown^2 w + \lambda\, g\, w = 0 \quad$ *on* Ω°, $\qquad w = 0 \quad$ *on* $\partial\Omega$,

where λ is some constant. It can be proved that a nontrivial solution w in general only exists for some values of λ, the so-called eigenvalues.

1. *Show by applying Green's first identity that the eigenvalues are positive.*

Assume that w and W are solutions of (3.2) for different *eigenvalues, such that*

$$\left.\begin{array}{l} \bigtriangledown^2 w + \lambda\, g\, w = 0 \\[2mm] \bigtriangledown^2 W + \Lambda\, g\, W = 0 \end{array}\right\} \quad på\ \Omega^\circ, \qquad \left.\begin{array}{c} w = 0 \\[2mm] W = 0 \end{array}\right\} \quad på\ \partial\Omega, \qquad \lambda \neq \Lambda.$$

2. *Show by applying Green's second identity that*

$$\int_\Omega g(\mathbf{x}\, w(\mathbf{x})\, W(\mathbf{x})\, d\Omega = 0.$$

We say that the functions w and W are orthogonal, *and g is called a* weight function.

A Eigenvalue problem; Green's first and second formulæ.

D Follow the guidelines.

I 1) Choose $g = g = w$ in Green's first formula. Then $w = 0$ on $\partial\Omega$ and

(7) $$\int_\Omega \left\{ w \bigtriangledown^2 w + \| \bigtriangledown f \|^2 \right\} d\Omega = \int_{\partial\Omega} w\, \frac{\partial w}{\partial n}\, dS = 0.$$

We have by (6 that $\bigtriangledown^2 w = -\lambda\, g\, w$, thus by a rearrangement of (7),

$$\int_\Omega \| \bigtriangledown w \|^2 d\Omega = -\int_\Omega w \bigtriangledown^2 w\, d\Omega = +\lambda \int_\Omega g \cdot w^2\, d\Omega.$$

since w is a non-trivial solution, we must have that $\bigtriangledown w \neq 0$ (w is not a constant), and

$$\int_\Omega \| \bigtriangledown w \|^2\, d\Omega > 0 \quad \text{and} \quad \int_\Omega g \cdot w^2\, d\Omega > 0,$$

hence

$$\lambda = \frac{\int_\Omega \| \bigtriangledown w \|^2\, d\Omega}{\int_\Omega g \cdot w^2\, d\Omega}$$

is defined and positive. It follows that every eigenvalue λ is positive.

2) Let w and W be non-trivial solutions for different eigenvalues λ and Λ. Then apply Green's second identity, using that w and W are zero on $\partial\Omega$,

$$\int_\Omega \left\{w \bigtriangledown^2 W - W \bigtriangledown^2 w\right\} d\Omega = \int_{\partial\Omega} \left\{w \frac{\partial W}{\partial n} - W \frac{\partial w}{\partial n}\right\} dS = 0.$$

Hence,

$$\int_\Omega w \bigtriangledown^2 W \, d\Omega - \int_\Omega W \bigtriangledown^2 w \, d\Omega = 0.$$

Since

$$\bigtriangledown^2 W = -\Lambda \, g \, W \qquad \text{og} \qquad \bigtriangledown^2 w = -\lambda \, g \, w,$$

it follows by insertion that

$$0 = -\int_\Omega w \, \Lambda \, g \, W \, d\Omega + \int_\Omega W \, \lambda \, g \, w \, d\Omega = (\lambda - \Lambda) \int_\Omega g \, w \, W \, d\Omega.$$

As $\lambda \neq \Lambda$, this implies

$$\int_\Omega g(\mathbf{x}) \, w(\mathbf{x}) \, W(\mathbf{x}) \, d\Omega = 0$$

as requested.

Example 3.3 *Let m be a constant. consider a function $f : \mathbb{R}^3 \to \mathbb{R}$, which satisfies*

$$f(\tau\,\mathbf{x}) = \tau^m\, f(\mathbf{x})$$

for every $\mathbf{x}$ and for those values of τ, for which τ^m is defined. We say that such a function is homogeneous *of degree m.*

1) Show that if f is also differentiable, then

$$\mathbf{x} \cdot \bigtriangledown f(\mathbf{x}) = m\, f(\mathbf{x}).$$

2) Show that if f furthermore is harmonic, then

$$\int_{K(\mathbf{x};a)} \| \bigtriangledown f\|^2\, d\Omega = \frac{m}{a} \int_{\partial \overline{K}(\mathbf{0};a)} f^2\, dS.$$

A Homogeneous functions of degree m.

D The first question follows by differentiation of the definition with respect to τ. In the second question we apply Green's first identity.

I 1) When we differentiate $f(\tau\,\mathbf{x}) = \tau^m\, f(\mathbf{x})$ with respect to τ, we get

$$m\,\tau^{m-1}\, f(\mathbf{x}) = \frac{d}{d\tau}\, f(\tau\,\mathbf{x}) = \mathbf{x} \cdot \bigtriangledown f(\tau\,\mathbf{x}).$$

Now, put $\tau = 1$. Then

$$\mathbf{x} \cdot \bigtriangledown f(\mathbf{x}) = m\, f(\mathbf{x}).$$

2) By Green's first identity,

$$\int_{\Omega} (g \bigtriangledown^2 f + \bigtriangledown g \cdot \bigtriangledown f)\, d\Omega = \int_{\partial\Omega} g\, \frac{\partial f}{\partial n}\, dS.$$

Choose $\Omega = K(\mathbf{0};a)$ and $g0f$. Since f is harmonic, $\bigtriangledown^2 f = 0$, it follows that

$$\int_{K(\mathbf{0};a)} \| \bigtriangledown f\|^2\, d\Omega = \int_{\partial \overline{K}(\mathbf{0};a)} f\, \frac{\partial f}{\partial n}\, dS.$$

We have on the sphere that $\mathbf{x} = a\,\mathbf{n}$, hence by 1),

$$\frac{\partial f}{\partial n} = \mathbf{n} \cdot \bigtriangledown f(\mathbf{x}) = \frac{1}{a}\, \mathbf{x} \cdot \bigtriangledown f(\mathbf{x}) = \frac{m}{a}\, f(\mathbf{x}).$$

$$\int_{K(\mathbf{0};a)} \| \bigtriangledown f\|^2\, d\Omega = \frac{m}{a} \int_{\partial \overline{K}(\mathbf{0};a)} f^2\, dS.$$

Remark. We strongly exploit that Ω is a ball of centrum $\mathbf{0}$. $\Diamond$

4 Curvilinear coordinates

Example 4.1 *Let $P_n(x,y,z)$ be a homogeneous polynomial of degree n. Then in spherical coordinates,*

$$\frac{\partial P_n}{\partial r} = \frac{n}{r}\,P_n, \qquad r \neq 0.$$

1) Prove this by first noticing that we can write

$$R_n(x,y,z) = r^n\,g(\theta,\varphi).$$

2) Show that if P_n is a harmonic function, then the function

$$Q_n(x,y,z) = \frac{P_n(x,y,z)}{r^{2n+1}}, \qquad r \neq 0,$$

is also harmonic.

A Homogeneous polynomial as an harmonic function in spherical coordinates.

D Follow the guidelines.

I 1) We have in spherical coordinates,

$$x^k y^\ell z^m = r^k\,g_1(\theta,\varphi)\cdot r^\ell\,g_2(\theta,\varphi)\cdot r^m\,g_3(\theta,\varphi) = r^{k+\ell+m} g_{k,\ell,m}(\theta,\varphi).$$

In an homogeneous polynomial all such terms satisfy

$$k+\ell+m = n \qquad (= \text{ the degree}),$$

thus by addition,

$$P_n(x,y,z) = r^n\,g(\theta,\varphi).$$

Hence for $r \neq 0$,

$$\frac{\partial P_n}{\partial r} = n\,r^{n-1}\,g(\theta,\varphi) = \frac{n}{r}\,r^n\,g(\theta,\varphi) = \frac{n}{r}\,P_n.$$

2) From

$$\bigtriangledown r = \frac{1}{r}\,\mathbf{x}, \qquad r \neq 0,$$

follows that

$$\bigtriangledown(r^\alpha) = \alpha\,r^{\alpha-1}\bigtriangledown r = \alpha\,r^{\alpha-2}\,\mathbf{x},$$

thus

$$\begin{aligned}\bigtriangledown^2(r^\alpha) &= \bigtriangledown\cdot\bigtriangledown(r^\alpha) = \bigtriangledown\cdot\left\{\alpha\,r^{\alpha-2}\,\mathbf{x}\right\} = \alpha(\alpha-2)r^{\alpha-4}\,\mathbf{x}\cdot\mathbf{x} + 3\alpha\,r^{\alpha-2}\\ &= \alpha(\alpha-2)\,r^{\alpha-4}\,r^2 + 3\alpha\,r^{\alpha-2} = \alpha(\alpha+1)r^{\alpha-2}.\end{aligned}$$

Let P_n be an homogeneous polynomial of degree n, which is also harmonic, i.e. $\bigtriangledown^2 P_n = 0$, and let

$$Q_n(x,y,z) = \frac{P_n(x,y,z)}{r^{2n+1}}, \qquad r \neq 0.$$

Choose $\alpha = -2n-1$. Then we get for $r \neq 0$ that

$$\begin{aligned}\bigtriangledown^2 Q_n &= \bigtriangledown \cdot \bigtriangledown \left(P_n \cdot r^{-2n-1}\right) = \bigtriangledown \cdot \left\{\bigtriangledown P_n \cdot r^{-2n-1} + P_n \bigtriangledown \left(r^{-2n-1}\right)\right\} \\ &= \bigtriangledown^2 P_n \cdot r^{-2n-1} + 2 \bigtriangledown P_n \cdot \bigtriangledown \left(r^{-2n-1}\right) + P_n \cdot \bigtriangledown^2 \left(r^{-2n-1}\right) \\ &= 2 \bigtriangledown P_n \cdot \mathbf{x}\,(-2n-1) r^{-2n-3} + P_n \cdot (-2n-1)(-2n) r^{-2n-3} \\ &= 2(2n+1) r^{-2n-3} \left\{- \bigtriangledown P_n \cdot \mathbf{x} + n\, P_n\right\}.\end{aligned}$$

If $k+\ell+m=n$, then

$$\begin{aligned}\bigtriangledown(x^k y^\ell z^m) \cdot (x,y,z) &= \left(k\, x^{k-1}, \ell\, y^{\ell-1}, m\, z^{m-1}\right) \cdot (x,y,z) \\ &= (k+\ell+m) x^k y^\ell z^m = n\, x^k y^\ell z^m.\end{aligned}$$

By adding all such terms we get

$$\bigtriangledown P_n \cdot \mathbf{x} = n\, P_n,$$

hence by insertion $\bigtriangledown^2 Q_n = 0$, and we have proved that Q_n is harmonic.

Remark. In the open octant, where $x > 0$, $y > 0$ and $z > 0$, the proof is carried over unchanged, even if k, ℓ, m and n are not integers. Another immediate extension is to negative integers, et.. $\Diamond$

Example 4.2 *We introduce the so-called* spheroidal *coordinates $(\eta, \vartheta, \varphi)$, where φ has the usual sense, by the following equations expressed in the rectangular coordinates,*

$$x = a\,\sinh\eta\,\sin\vartheta\,\cos\varphi, \qquad y = a\,\sinh\eta\,\sin\vartheta\,\sin\varphi, \qquad z = a\,\cosh\eta\,\cos\vartheta.$$

1) Describe the coordinate surfaces and find the intervals of η and ϑ.

2) Show that $(\eta, \vartheta, \varphi)$ are orthogonal.

3) Find the metric coefficients.

4) Show that the function $f(\eta,\vartheta,\varphi) = \ln\tanh\left(\dfrac{\eta}{2}\right)$ is a solution of Laplace's equation.

A Spheroidal coordinates.

D Apply the description on any given textbook.

I 1) Let $\eta \neq 0$ be fixed. Then it follows from

$$x^2 + y^2 = a^2 \sinh^2\eta\, \sin^2\vartheta, \qquad z^2 = a^2 \cosh^2\eta\, \cos^2\vartheta,$$

that

$$\frac{x^2+y^2}{(a\,\sinh\eta)^2} + \frac{z^2}{(a\,\cosh\eta)^2} = 1.$$

This equation describes an ellipsoidal surface of the half axes $a|\sinh\eta|$, $a|\sinh\eta|$ and $a\cosh\eta$. Notice that we obtain the same ellipsoid, when η is replaced by $-\eta$, and with the exception of the points on the segment $[-a,a]$ of the Z-axis, every point in space lies on precisely one such ellipsoidal surface, where $\eta > 0$.

Roughly speaking, this means that the ellipsoidal surface is inflated *continuously* like a balloon, when the parameter $\eta > 0$ increases.

If $\eta = 0$, then $x = 0$, $y = 0$ and $z = a\,\cos\vartheta$, which describes the segment $[-a, a]$ on the Z-axis run through once if $\vartheta \in [0, \pi]$.

We hereby obtain the level surfaces $\eta > 0$ where $\eta = 0$ is the generated case, and the η-interval is $[0, +\infty[$.

It also follows from the above that $\vartheta \in [0, \pi]$.

Now notice that if $\vartheta = 0$, then $x = 0$, $y = 0$ and $z = a\,\cosh\eta$, which for $\eta \geq 0$ describes the half line $[a, +\infty[$ on the Z-axis run through once.

If $\vartheta \in \left]0, \dfrac{\pi}{2}\right[$ is fixed, then $z > 0$, and we get by eliminating η and φ,

$$(8)\quad \frac{z^2}{(a\,\cos\vartheta)^2} - \frac{x^2+y^2}{a\,\sin\vartheta)^2} = 1,$$

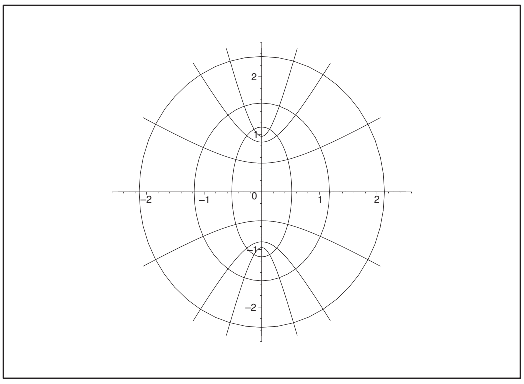


Figure 11: The meridian curves extended to the whole plane for $a = 1$.

which describes the upper net of an hyperboloid of two nets and "half axes" $a|\sin\vartheta|$, $a|\sin\vartheta|$ and $a\cos\vartheta$, where we have used absolute values to support the following, although this is not necessary.

If $\vartheta \in \left]\frac{\pi}{2},\pi\right[$ is fixed, then $z < 0$. When we eliminate η and φ we again obtain (8), and we get the corresponding lower net as level surfaces.

If $\vartheta = \frac{\pi}{2}$, then

$$x = a\,\sinh\eta\,\cos\varphi, \quad y = a\,\sinh\eta\,\sin\varphi, \quad z = 0,$$

which (put $\varrho = a\,\sinh\eta$) runs through the plane $z = 0$, so this is the level surface of $\vartheta = \frac{\pi}{2}$.

If $\vartheta = \pi$, then $x = 0$, $y = 0$ and $z = -a\,\cosh\eta$, which describes the half line $]-\infty,-a[$ on the Z-axis.

By a continuity argument it follows that every point in space lies precisely on one of these level surfaces (degenerated for $\vartheta = 0$ and $\vartheta = \pi$).

If $\varphi \in [0,2\pi[$ is kept fixed, we get a meridian half plane, when $\eta \geq 0$ and $\vartheta \in [0,\pi]$ vary.

As a conclusion we have described the level surfaces,and the intervals are

$$\eta \in [0,+\infty[, \qquad \vartheta \in [0,\pi], \qquad \varphi \in [0,2\pi[.$$

2) Clearly, the meridian half planes are orthogonal to the other level surfaces.

Until 40 years ago it was even known in high school that the hyperbolic system and the elliptic system are orthogonal. This may perhaps no longer be the case. Instead we get by a computation

$$(9)\quad \begin{cases} \dfrac{\partial \mathbf{r}}{\partial \eta} = a(\cosh\eta\,\sin\vartheta\,\cos\varphi, \cosh\eta\,\sin\vartheta\,\sin\varphi, \sinh\eta\,\cos\vartheta), \\[2ex] \dfrac{\partial \mathbf{r}}{\partial \vartheta} = a(\sinh\eta\,\cos\vartheta\,\cos\varphi, \sinh\eta\,\cos\vartheta\,\sin\varphi, -\cosh\eta\,\sin\vartheta), \\[2ex] \dfrac{\partial \mathbf{r}}{\partial \varphi} = a(-\sinh\eta\,\sin\vartheta\,\sin\varphi, \sinh\eta\,\sin\vartheta\,\cos\varphi, 0). \end{cases}$$

Hence

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \eta}\cdot\frac{\partial \mathbf{r}}{\partial \vartheta} &= a^2\{\sinh\eta\,\cosh\eta\,\sin\vartheta\,\cos\vartheta(\cos^2\varphi+\sin^2\varphi) - \sinh\eta\,\cosh\eta\,\sin\vartheta\,\cos\vartheta\} \\ &= 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \eta}\cdot\frac{\partial \mathbf{r}}{\partial \varphi} &= a^2\{-\sinh\eta\,\cosh\eta\sin^2\vartheta\,\sin\varphi\cos\varphi + \sinh\eta\,\cosh\eta\,\sin^2\vartheta\,\sin\varphi\,\cos\varphi\} \\ &= 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \eta}\cdot\frac{\partial \mathbf{r}}{\partial \varphi} &= a^2\{-\sinh^2\eta\,\sin\vartheta\,\cos\vartheta\,\sin\varphi\,\cos\varphi + \sinh^2\eta\,\sin\vartheta\,\cos\vartheta\,\sin\varphi\,\cos\varphi\} \\ &= 0. \end{aligned}$$

We have now proved that $(\eta, \vartheta, \varphi)$ are orthogonal.

3) It follows from (9) that

$$\begin{aligned} h_1^2 &= \frac{\partial \mathbf{r}}{\partial \eta}\cdot\frac{\partial \mathbf{r}}{\partial \eta} \\ &= a^2\left\{\cosh^2\eta\,\sin^2\vartheta\,\cos^2\varphi + \cosh^2\eta\,\sin^{@}\vartheta\,\sin^2\varphi + \sinh^2\eta\,\cos^2\vartheta\right\} \\ &= a^2\left\{\cosh^2\eta\,\sin^2\vartheta + \sinh^2\eta\,\cos^2\vartheta\,\cos^2\vartheta\right\} = a^2\left\{\sin h^2\eta + \sin^2\vartheta\right\} \\ &= a^2\left\{\cosh^2\eta - \cos^2\vartheta\right\}, \end{aligned}$$

$$\begin{aligned} h_2^2 &= \frac{\partial \mathbf{r}}{\partial \vartheta}\cdot\frac{\partial \mathbf{r}}{\partial \vartheta} \\ &= a^2\left\{\sinh^2\eta\,\cos^2\vartheta\cos^2\varphi + \sinh^2\eta\,\cos^2\vartheta\,\sin^2\varphi + \cosh^2\eta\,\sin^2\vartheta\right\} \\ &= a^2\left\{\sinh^2\eta\,\cos^2\vartheta + \cosh^2\eta\,\sin^2\vartheta\right\} = h_1^2 = a^2\left\{\sinh^2\eta + \sin^2\vartheta\right\} \\ &= a^2\left\{\cosh^2\eta - \cos^2\vartheta\right\}, \end{aligned}$$

$$\begin{aligned} h_3^2 &= \frac{\partial \mathbf{r}}{\partial \varphi}\cdot\frac{\partial \mathbf{r}}{\partial \varphi} \\ &= a^2\left\{\sinh^2\eta\,\sin^2\vartheta\,\sin^2\varphi + \sinh^2\eta\,\sin^2\vartheta\,\cos^2\varphi\right\} \\ &= a^2\sinh^2\eta\,\sinh^2\vartheta. \end{aligned}$$

For $\eta \geq 0$ and $0 \leq \vartheta \leq \pi$ the metric coefficients are

$$h_1 = h_2 = a\sqrt{\sinh^2\eta + \sin^2\vartheta} \quad \text{and} \quad h_3 = a\,\sinh\eta\,\sin\vartheta.$$

4) Since $f(\eta,\vartheta,\varphi)=\ln\tanh\left(\dfrac{\eta}{2}\right)$ is independent of ϑ and φ, we have

$$\bigtriangledown f=\frac{1}{h_1}\,\frac{\partial f}{\partial\eta}\,\mathbf{a}_1+\frac{1}{h_2}\,\frac{\partial f}{\partial\vartheta}\,\mathbf{a}_2+\frac{1}{h_3}\,\frac{\partial f}{\partial\varphi}\,\mathbf{a}_3=\frac{1}{h_1}\,\frac{\partial f}{\partial\eta}\,\mathbf{a}_1=\frac{1}{h_1}\cdot\frac{1}{\sinh\eta}\,\mathbf{a}_1,$$

because

$$\frac{\partial f}{\partial\eta}=\frac{1}{\tanh\left(\frac{\eta}{2}\right)}\cdot\frac{1}{\cosh^2\left(\frac{\eta}{2}\right)}\cdot\frac{1}{2}=\frac{1}{2\sinh\left(\frac{\eta}{2}\right)\cosh\left(\frac{\eta}{2}\right)}=\frac{1}{\sinh\eta}.$$

Hence, the coordinates are with respect to the new system,

$$(V_1,V_2,V_3)=\left(\frac{1}{h_1\sinh\eta}\,,\,0\,,\,0\right),$$

thus

$$\begin{aligned}
\bigtriangledown^2 f &= \bigtriangledown\cdot\bigtriangledown f\\
&= \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial\eta}\,(h_2h_3V_1)+\frac{\partial}{\partial\vartheta}\,(h_3h_1V_2)+\frac{\partial}{\partial\varphi}\,(h_1h_2V_3)\right\}\\
&= \frac{1}{h_1h_2h_3}\,\frac{\partial}{\partial\eta}\left(\frac{h_2h_3}{h_1}\cdot\frac{1}{\sinh\eta}\right)\\
&= \frac{1}{h_1h_2h_3}\,\frac{\partial}{\partial\eta}\left(\frac{a\,\sinh\eta\,\sin\vartheta}{\sinh\eta}\right)=\frac{1}{h_1h_2h_3}\,\frac{\partial}{\partial\eta}\,\sin\vartheta=0,
\end{aligned}$$

and the claim is proved.

Remark. By a similar argument we see that

$$g(\eta,\vartheta,\varphi)=\frac{1}{2}\,\ln\left(\frac{1-\cos\vartheta}{1+\cos\vartheta}\right),\qquad 0<\vartheta<\pi,$$

satisfies Laplace's equation. $\Diamond$

Example 4.3 *Let (u,v,z) be an orthogonal cylinder coordinate system, the metric coefficients of which satisfy $h_u=h_v$. Prove that the functions*

$$F(u,v,z)=\alpha+\beta u,\qquad G(u,v,z)=e^{\gamma u}\cos(\gamma v),$$

α, β, γ being known constants, satisfy Laplace's equation.
Find more similar solutions of Laplace's equation.

A Orthogonal cylinder coordinate system. Laplace's equation.

D Apply the Laplace operator.

I Since the cylinder coordinate system is orthogonal, we have $h_z=1$. When we set up the Laplace operator it then becomes a question of making the right identificcations:

$$h_1=h_2=h_u=h_v\qquad\text{and}\qquad h_3=1,$$

thus

$$
\begin{aligned}
\nabla^2 f &= \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\,\frac{\partial f}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{h_3h_1}{h_2}\,\frac{\partial f}{\partial v}\right)+\frac{\partial}{\partial z}\left(\frac{h_1h_2}{h_3}\,\frac{\partial f}{\partial z}\right)\right\}\\
&= \frac{1}{h_u^2}\left\{\frac{\partial}{\partial u}\left(\frac{h_u}{h_u}\,\frac{\partial f}{\partial u}+\frac{h_u}{h_u}\,\frac{\partial f}{\partial v}\right)+\frac{\partial}{\partial z}\left(\frac{h_u^2}{1}\,\frac{\partial f}{\partial z}\right)\right\}\\
&= \frac{1}{h_u^2}\left\{\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}+\frac{\partial}{\partial z}\left(h_u^2\,\frac{\partial f}{\partial z}\right)\right\}.
\end{aligned}
$$

Any function, which only depends on u and v, and which is harmonic in these variables, must satisfy the Laplace equation. This is trivial for $F(u,v,z)$, which is a polynomial of degree 1. Furthermore,

$$\frac{\partial^2 G}{\partial u^2}+\frac{\partial^2 G}{\partial v^2}=\gamma^2 G(u,v,z)-\gamma^2 G(u,v,z)=0,$$

so the claim also holds for $G(u,v,z)$.

As mentioned above, any harmonic function in (u,v) and independent of z satisfies Laplace's equation.

REMARK 1. With the knowledge of Complex Function Theory one obtains the harmonic functions by taking the real part or the imaginary part of an analytic function. $\Diamond$

REMARK 2. If we do not know $h_u = h_v$, it is not possible to find the solutions of Laplace's equation which also depend on z. $\Diamond$

Example 4.4 *Assume that the constants α, β satisfy $\beta > \alpha > 0$. Consider for fixed (x, y, z), where $xyz \neq 0$, the auxiliary function*

$$F(t) = x^2(t-\alpha)(t-\beta)+y^2t(t-\beta)+z^2t(t-\alpha)-t(t-\alpha)(t-\beta), \quad t \in \mathbb{R}.$$

1. *Find $F(0)$, $F(\alpha)$ and $F(\beta)$. Then sketch the graph of F and show that the equation $F(t) = 0$ has three different roots u, v, w, which satisfy*

$$0 < u < \alpha < v < \beta < w.$$

Hence, for every (x, y, z) with $xyz \neq 0$ we obtain a set (u, v, w). These are called the ellipsoidal coordinates *of the point with respect to the constants α and β.*

2. *Show that the coordinate surfaces (in the (x, y, z)-space) are parts of the following surfaces:*

- *ellipsoids for w constant,*
- *hyperboloids with 1 net for v constant, and*
- *hyperboloids with 2 nets for u constant.*

3. *Show that*

$$F(t) = (u-t)(v-t)(w-t),$$

and then derive the expressions

$$x^2 = \frac{uvw}{\alpha\beta}, \qquad y^2 = \frac{(\alpha-u)(v-\alpha)(w-\alpha)}{\alpha(\beta-\alpha)}, \qquad z^2 = \frac{(\beta-u)(\beta-v)(w-\beta)}{\beta(\beta-\alpha)}.$$

Thus, the transformation to the new coordinates is not injective.

4. *Show that the new coordinate system is orthogonal.*

5. *Show that the metric coefficients are given by*

$$h_u = \frac{1}{2}\sqrt{\frac{(v-u)(w-u)}{u(\alpha-u)(\beta-u)}},$$

$$h_v = \frac{1}{2}\sqrt{\frac{(v-u)(w-v)}{v(v-\alpha)(\beta-v)}},$$

$$h_w = \frac{1}{2}\sqrt{\frac{(w-v)(w-u)}{w(w-\alpha)(w-\beta)}}.$$

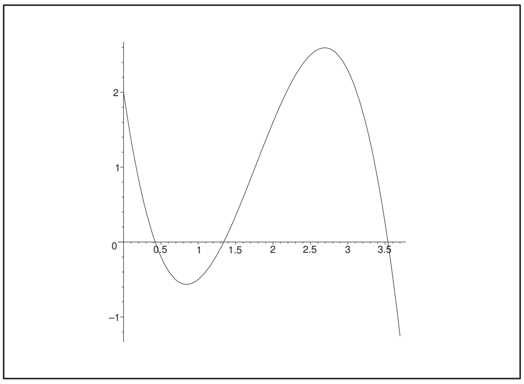


Figure 12: the graph of $F(t)$, when $x = 1$, $y = 0,5$, $z = 0,8$ and $\alpha = 1$, $\beta = 2$.

A Curvilinear coordinates.

D Apply the theory.

I 1) Clearly, $F(t)$ is a polynomial of third degree and

$$F(0) = \alpha\beta x^2 > 0, \quad F(\alpha) = -\alpha(\beta-\alpha)y^2 < 0, \quad F(\beta) = \beta(\beta-\alpha)z^2 > 0.$$

Also,

$$\lim_{t\to-\infty} F(t) = +\infty \quad \text{og} \quad \lim_{t\to+\infty} F(t) = -\infty.$$

Since $F(t)$ is continuous we conclude from the variation of the signs, cf. the figure, that there are three different roots u, v, w, which satisfy

$$0 < u < \alpha < v < \beta < w.$$

For each (x, y, z) where $xyz \neq 0$ we have precisely one such set (u, v, w).

Remark. It follows that $(\pm x, \pm y, \pm z)$ with $xyz \neq 0$ for each of the eight possible choices of the signs give the same set (u, v, w), so the transformation is not injective, cf. 3). $\Diamond$

2) a) When $t = w$, we get $F(w) = 0$, hence by a rearrangement

$$(w-\alpha)(w-\beta)x^2 + w(w-\beta)y^2 + w(w-\alpha)z^2 = w(w-\alpha)(w-\beta).$$

From $w > \beta > \alpha > 0$ follows that all coefficients are positive, hence we see by a continuous extension to the coordinate planes $x = 0$, $y = 0$ and $z = 0$ that the coordinate surface is an ellipsoid.

b) Similarly, we get for $t = v$ that $F(v) = 0$. As $0 < \alpha < v < \beta$, it follows by a rearrangement that

$$(v-\alpha)(v-\beta)x^2 + v(v-\beta)y^2 + v(v-\alpha)z^2 = v(v-\alpha)(v-\beta).$$

Hence by a change of signs, such that all terms are positive, with the exception of $-v(v-\alpha)$,

$$(v-\alpha)(\beta-v)x^2 + v(\beta-v)y^2 - v(v-\alpha)z^2 = v(v-\alpha)(\beta-v),$$

corresponding to that (the continuous extension of) the coordinate surface is an hyperboloid with 1 net.

c) As $t = u$ gives $F(u) = 0$, where $0 < u < \alpha < \beta$, it follows by a rearrangement that

$$(u-\alpha)(u-\beta)x^2 + u(u-\beta)y^2 + u(8-\alpha)z^2 = u(u-\alpha)(u-\beta),$$

corresponding to that (the continuous extension of) the coordinate surface is an hyperboloid of 2 nets. (The right hand side is positive and the coefficient of x^2 is positive, while the coefficients of y^2 and z^2 are both negative).

3) Since $F(t)$ is a polynomial of degree 3 with the coefficient -1 of t^3, and if u, v, w are the three roots, then

$$F(t) = -(t-u)(t-v)(t-w) = (u-t)(v-t)(w-t).$$

It follows from 1) and this alternative description that

$$F(0) = \alpha\beta x^2 = uvw, \qquad \text{hence } x^2 = \frac{uvw}{\alpha\beta},$$

and

$$F(\alpha) = -\alpha(\beta-\alpha)y^2 = (u-\alpha)(v-\alpha)(w-\alpha),$$

hence

$$y^2 = \frac{(\alpha-u)(v-\alpha)(w-\alpha)}{\alpha(\beta-\alpha)},$$

and

$$F(\beta) = \beta(\beta-\alpha)z^2 = (u-\beta)(v-\beta)(w-\beta),$$

and thus

$$z^2 = \frac{(u-\beta)(v-\beta)(w-\beta)}{\beta(\beta-\alpha)} = \frac{(\beta-u)(\beta-v)(w-\beta)}{\beta(\beta-\alpha)}.$$

4) It follows from the results of 3) that

$$\frac{\partial(x^2)}{\partial u} = 2x\,\frac{\partial x}{\partial u} = \frac{vw}{\alpha\beta},$$

thus

$$\frac{\partial x}{\partial u} = \frac{1}{2}\,\mathrm{sign}(x)\sqrt{\frac{\alpha\beta}{uvw}}\cdot\frac{vw}{\alpha\beta} = \frac{\mathrm{sign}(x)}{2\sqrt{\alpha\beta}}\sqrt{\frac{vw}{u}}.$$

Due to the symmetry we can interchange the letters, which gives

$$\frac{\partial x}{\partial v} = \frac{\mathrm{sign}(x)}{2\sqrt{\alpha\beta}}\sqrt{\frac{uw}{v}}, \qquad \frac{\partial x}{\partial w} = \frac{\mathrm{sign}(x)}{2\sqrt{\alpha\beta}}\sqrt{\frac{uv}{w}}.$$

Furthermore,

$$\frac{\partial(y^2)}{\partial u} = 2y\,\frac{\partial y}{\partial u} = -\frac{(v-\alpha)(w-\alpha)}{\alpha(\beta-\alpha)},$$

hence

$$\frac{\partial y}{\partial u} = -\frac{\operatorname{sign}(y)}{2\sqrt{\alpha(\beta-\alpha)}}\sqrt{\frac{(v-\alpha)(w-\alpha)}{\alpha-u}},$$

and similarly (NB: change of sign!)

$$\frac{\partial y}{\partial v} = \frac{\operatorname{sign}(y)}{2\sqrt{\alpha(\beta-\alpha)}}\sqrt{\frac{(\alpha-u)(w-\alpha)}{v-\alpha}},$$

$$\frac{\partial y}{\partial w} = \frac{\operatorname{sign}(y)}{2\sqrt{\alpha(\beta-\alpha)}}\sqrt{\frac{(\alpha-u)(v-\alpha)}{w-\alpha}}.$$

Finally, we get in exactly the same way,

$$\frac{\partial z}{\partial u} = -\frac{\operatorname{sign}(z)}{2\sqrt{\beta(\beta-\alpha)}}\sqrt{\frac{(\beta-v)(w-\beta)}{\beta-u}},$$

$$\frac{\partial z}{\partial v} = -\frac{\operatorname{sign}(z)}{2\sqrt{\beta(\beta-\alpha)}}\sqrt{\frac{(\beta-u)(w-\beta)}{\beta-v}},$$

$$\frac{\partial z}{\partial w} = \frac{\operatorname{sign}(z)}{2\sqrt{\beta(\beta-\alpha)}}\sqrt{\frac{(\beta-u)(\beta-v)}{w-\beta}}.$$

According to the theory,

$$d\mathbf{x} = h_u\mathbf{a}_u du + h_v\mathbf{a}_v dv + h_w\mathbf{a}_w dw,$$

where it follows from the above that

$$h_u\mathbf{a}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \frac{1}{2}\begin{pmatrix} \dfrac{\operatorname{sign}(x)}{\sqrt{\alpha\beta}}\sqrt{\dfrac{vw}{u}} \\[3mm] -\dfrac{\operatorname{sign}(y)}{\sqrt{\alpha(\beta-\alpha)}}\sqrt{\dfrac{(v-\alpha)(w-\alpha)}{\alpha-u}} \\[3mm] -\dfrac{\operatorname{sign}(z)}{\sqrt{\beta(\beta-\alpha)}}\sqrt{\dfrac{(\beta-v)(w-\beta)}{\beta-u}} \end{pmatrix},$$

$$h_v\mathbf{a}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \frac{1}{2}\begin{pmatrix} \dfrac{\operatorname{sign}(x)}{\sqrt{\alpha\beta}}\sqrt{\dfrac{uw}{v}} \\[3mm] \dfrac{\operatorname{sign}(y)}{\sqrt{\alpha(\beta-\alpha)}}\sqrt{\dfrac{(\alpha-u)(w-\alpha)}{v-\alpha}} \\[3mm] -\dfrac{\operatorname{sign}(z)}{\sqrt{\beta(\beta-\alpha)}}\sqrt{\dfrac{(\beta-u)(w-\beta)}{\beta-v}} \end{pmatrix},$$

$$h_w\mathbf{a}_w = \left(\frac{\partial x}{\partial w}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial w}\right) = \frac{1}{2}\begin{pmatrix} \dfrac{\operatorname{sign}(x)}{\sqrt{\alpha\beta}}\sqrt{\dfrac{uv}{w}} \\[3mm] \dfrac{\operatorname{sign}(y)}{\sqrt{\alpha(\beta-\alpha)}}\sqrt{\dfrac{(\alpha-u)(v-\alpha)}{w-\alpha}} \\[3mm] \dfrac{\operatorname{sign}(z)}{\sqrt{\beta(\beta-\alpha)}}\sqrt{\dfrac{(\beta-u)(\beta-v)}{w-\beta}} \end{pmatrix}.$$

Thus

$$\begin{aligned} h_u\mathbf{a}_u \cdot h_v\mathbf{a}_v &= \frac{1}{4}\left\{\frac{w}{\alpha\beta} - \frac{w-\alpha}{\alpha(\beta-\alpha)} + \frac{w-\beta}{\beta(\beta-\alpha)}\right\} \\ &= \frac{1}{4}\left\{w\cdot\frac{\beta-\alpha-\beta+\alpha}{\alpha\beta(\beta-\alpha)} + \frac{\alpha}{\alpha(\beta-\alpha)} - \frac{\beta}{\beta(\beta-\alpha)}\right\} = 0. \qquad (10) \end{aligned}$$

$$h_u\mathbf{a}_u \cdot h_w\mathbf{a}_w = \frac{1}{4}\left\{\frac{v}{\alpha\beta} - \frac{v-\alpha}{\alpha(\beta-\alpha)} + \frac{v-\beta}{\beta(\beta-\alpha)}\right\} = 0,$$

$$h_v\mathbf{a}_v \cdot h_w\mathbf{a}_w = \frac{1}{4}\left\{\frac{u}{\alpha\beta} - \frac{u-\alpha}{\alpha(\beta-\alpha)} + \frac{u-\beta}{\beta(\beta-\alpha)}\right) = 0,$$

where we in the latter two calculations have used that they have the same structure as in (10), only with v or u instead of w.

Since $h_u h_v h_w \neq 0$, we conclude from the above that

$$\mathbf{a}_u \cdot \mathbf{a}_v = 0, \qquad \mathbf{a}_u \cdot \mathbf{a}_w = 0, \qquad \mathbf{a}_v \cdot \mathbf{a}_w = 0,$$

and the new coordinate system is orthogonal.

5) Finally, we derive from the results of 4) that we get (11), i.e.

$$\begin{aligned}
h_u^2 &= \left\|\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)\right\|^2 = h_u\mathbf{a}_u \cdot h_u\mathbf{a}_u \\
&= \frac{1}{4}\left\{\frac{1}{\alpha\beta}\,\frac{vw}{u} + \frac{1}{\alpha(\beta-\alpha)}\,\frac{(v-\alpha)(w-\alpha)}{\alpha-u} + \frac{1}{\beta(\beta-\alpha)}\,\frac{(\beta-v)(w-\beta)}{\beta-u}\right\} \\
&= \frac{1}{4}\,\frac{1}{u(\alpha-u)(\beta-u)}\left\{\frac{vw}{\alpha\beta}(\alpha-u)(\beta-u) + \frac{(v-\alpha)(w-\alpha)}{\alpha(\beta-\alpha)}\,u(\beta-u) + \frac{(\beta-v)(w-\beta)}{\beta(\beta-\alpha)}\,u(\alpha-u)\right\}.
\end{aligned}$$

Here

$$\frac{vw}{\alpha\beta}(\alpha-u)(\beta-u) = \frac{u^2vw}{\alpha\beta} - \frac{\alpha+\beta}{\alpha\beta}\,uvw + vw,$$

$$\frac{(v-\alpha)(w-\alpha)}{\alpha(\beta-\alpha)}\,u(\beta-u) = -\frac{u^2vw}{\alpha(\beta-\alpha)} + \frac{\beta}{\alpha(\beta-\alpha)}\,uvw - \frac{\alpha(v+w)u(\beta-u)}{\alpha(\beta-\alpha)} + \frac{\alpha^2u(\beta-u)}{\alpha(\beta-\alpha)},$$

$$\frac{(\beta-v)(w-\beta)}{\beta(\beta-\alpha)} \cdot u(\alpha-u) = \frac{u^2vw}{\beta(\beta-\alpha)} - \frac{\alpha}{\beta(\beta-\alpha)}\,uvw + \frac{\beta(v+w)u(\alpha-u)}{\beta(\beta-\alpha)} - \frac{\beta^2u(\alpha-u)}{\beta(\beta-\alpha)},$$

so the latter factor $\{\cdots\}$ of (11) can now be written as

$$\begin{aligned}
&u^2vw\left\{\frac{1}{\alpha\beta} - \frac{1}{\alpha(\beta-\alpha)} + \frac{1}{\beta(\beta-\alpha)}\right\} + uvw\left\{-\frac{\alpha+\beta}{\alpha\beta} + \frac{\beta}{\alpha(\beta-\alpha)} - \frac{\alpha}{\beta(\beta-\alpha)}\right\} \\
&\qquad + vw + \frac{u(v+w)}{\beta-\alpha}\,\{-(\beta-u)+(\alpha-u)\} + u\left\{\frac{\alpha\beta}{\beta-\alpha} - \frac{\alpha\beta}{\beta-\alpha}\right\} + u^2\left\{-\frac{\alpha}{\beta-\alpha} + \frac{\beta}{\beta-\alpha}\right\} \\
&= u^2vw\left\{\frac{1}{\alpha\beta} - \frac{\beta}{\alpha\beta(\beta-\alpha)} + \frac{\alpha}{\alpha\beta(\beta-\alpha)}\right\} + uvw\left\{-\frac{\alpha+\beta}{\alpha\beta} + \frac{\beta^2-\alpha^2}{\alpha\beta(\beta-\alpha)}\right\} + vw \\
&\qquad + \frac{u(v+w)}{\beta-\alpha}\,(\alpha-\beta) + u^2 \\
&= u^2 - (v+w)u + vw = (u-v)(u-w) = (v-u)(w-u),
\end{aligned}$$

which by insertion into (11) gives

$$h_u^2 = \frac{1}{4} \cdot \frac{(v-u)(w-u)}{u(\alpha-u)(\beta-u)}$$

and we have proved that

$$0 < h_u = \frac{1}{2}\sqrt{\frac{(v-u)(w-u)}{u(\alpha-u)(\beta-u)}}.$$

By comparison of the expressions of $h_u\mathbf{a}_u$, $h_v\mathbf{a}_v$ and $h_w\mathbf{a}_w$, we see that there is some form of symmetry:
Put u into the denominator of $h_u\mathbf{a}_u$; similarly, put v in the denominator of $h_v\mathbf{a}_v$, and put w into the denominator of $h_w\mathbf{a}_w$. Thus, we *almost* get the expressions of h_v and h_w by interchanging the letters. The only additional complication is that all the factors shall be positive. Taking also this into account we finally get

$$h_v = \frac{1}{2}\sqrt{\frac{(v-u)(w-v)}{v(v-\alpha)(\beta-v)}} \quad \text{og} \quad h_w = \frac{1}{2}\sqrt{\frac{(w-v)(w-u)}{w(w-\alpha)(w-\beta)}}.$$

ALTERNATIVELY, just repeat the computations above. One immediately gets (11) by interchanging the letters u, v, w, so in the remaining part of the argument we shall only identify the coefficients (functions of α and β) in a polynomial in u, v, w.

Example 4.5 *Here we construct a variant of the spherical coordinates (r,θ,φ) by putting*

$$r = a\,e^{\xi}, \qquad \xi \in \mathbb{R},$$

while θ and φ are kept as previously. Clearly, the new system (ξ,θ,φ) is orthogonal. Find its metric coefficients h_ξ, h_θ, h_φ.

A Curvilinear coordinates.

D Write the rectangular coordinates in the new ones via the usual spherical coordinates, and then compute the metric coefficients.

I It is well-known that

$$\begin{aligned} x &= r\,\sin\theta\,\cos\varphi &&= a\,e^{\xi}\sin\theta\,\cos\varphi, \\ y &= r\,\sin\theta\,\sin\varphi &&= a\,e^{\xi}\sin\theta\,\sin\varphi, \\ z &= r\,\cos\theta &&= a\,e^{\xi}\cos\theta. \end{aligned}$$

Hence

$$\frac{\partial\mathbf{r}}{\partial\xi} = \mathbf{r},$$

$$\frac{\partial\mathbf{r}}{\partial\theta} = a\,e^{\xi}(\cos\theta\,\cos\varphi, \cos\theta\,\sin\varphi, -\sin\theta),$$

$$\frac{\partial\mathbf{r}}{\partial\varphi} = a\,e^{\xi}\sin\theta(-\sin\varphi, \cos\varphi, 0),$$

and thus

$$\begin{aligned} h_\xi &= \left\| \frac{\partial \mathbf{r}}{\partial \xi} \right\| = \|\mathbf{r}\| = a\, e^{\xi}, \\ h_\theta &= \left\| \frac{\partial \mathbf{r}}{\partial \theta} \right\| = a\, e^{\xi}, \\ h_\varphi &= \left\| \frac{\partial \mathbf{r}}{\partial \varphi} \right\| = a\, e^{\xi} \sin\theta. \end{aligned}$$

Example 4.6 *The so-called* six ball coordinates (u,v,w) *are introduced in the following way:*

$$u = \frac{x}{r^2}, \qquad v = \frac{y}{r^2}, \qquad w = \frac{z}{r^2}.$$

1) Describe the coordinate surfaces.

2) Find $u^2+v^2+w^2$, and then express (x,y,z) by means of (u,v,w).

3) Show that the coordinate system (u,v,w) is orthogonal.

4) Compute the metric coefficients h_u, h_v, h_w.

A Curvilinear coordinates in $\mathbb{R}^3 \setminus \{(0,0,0)\}$.

D Identify each concept. We shall everywhere not consider the point $(x,y,z)=(0,0,0)$.

I 1) When $u=0$, we get $x=0$ (a plane).
When $u \neq 0$ is constant, then

$$0 = r^2 - \frac{x}{u} = x^2 - 2 \cdot \frac{1}{2u} \cdot x + \frac{1}{4u^2} + y^2 + z^2 - \frac{1}{4u^2},$$

hence

$$\left(x - \frac{1}{2u} \right)^2 + y^2 + z^2 = \left(\frac{1}{|2u|} \right)^2.$$

The coordinate surface corresponding to $u \neq 0$ constant is the sphere of centrum $\left(\frac{1}{2u}\,,\,0\,,\,0 \right)$ and radius $\frac{1}{|2u|}$.

Similarly, we get for $v=0$ the plane $y=0$, and for $v \neq 0$ constant we get

$$x^2 + \left(y - \frac{1}{2v} \right)^2 + z^2 = \left(\frac{1}{|2v|} \right)^2,$$

i.e. the sphere of centrum $\left(0\,,\,\frac{1}{2v}\,,\,0 \right)$ and radius $\frac{1}{|2v|}$.

Finally, $w = 0$ corresponds to the plane $z = 0$, and when $w \neq 0$ is a constant we get

$$x^2 + y^2 + \left(z - \frac{1}{2w}\right)^2 = \left(\frac{1}{|2w|}\right)^2,$$

thus the sphere of centrum $\left(0\,,\,0\,,\,\dfrac{1}{2w}\right)$ and radius $\dfrac{1}{|2w|}$.

2) A small computation gives

$$u^2 + v^2 + w^2 = \frac{1}{r^4}\left(x^2 + y^2 + z^2\right) = \frac{r^2}{r^4} = \frac{1}{r^2},$$

thus

$$r^2 = \frac{1}{u^2 + v^2 + w^2}.$$

Then

$$x = r^2 u = \frac{u}{u^2 + v^2 + w^2}, \qquad y = r^2 v = \frac{v}{u^2 + v^2 + w^2}, \qquad z = r^2 w = \frac{w}{u^2 + v^2 + w^2}.$$

3) It follows that

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u} &= \left(\frac{1}{u^2+v^2+w^2}-\frac{2u^2}{(u^2+v^2+w^2)^2},\frac{-2uv}{(u^2+v^2+w^2)^2},\frac{-2uw}{(u^2+v^2+w^2)}\right)\\ &= \frac{1}{(u^2+v^2+w^2)^2}\left(-u^2+v^2+w^2,-2uv,-2uw\right),\end{aligned}$$

and similarly,

$$\frac{\partial \mathbf{r}}{\partial v}=\frac{1}{(u^2+v^2+w^2)^2}\,(-2uv,u^2-v^2+w^2,-2vw),$$

$$\frac{\partial \mathbf{r}}{\partial w}=\frac{1}{(u^2+v^2+w^2)^2}\,(-2uw,-2vw,u^2+v^2-w^2).$$

Hence

$$\begin{aligned}(u^2+v^2+w^2)^4\,\frac{\partial \mathbf{r}}{\partial u}\cdot\frac{\partial \mathbf{r}}{\partial v} &= -2uv(-u^2+v^2+w^2)-2uv(u^2-v^2+w^2)+4uvw^2\\ &= 2uv\left\{-2w^2+2w^2\right\}=0,\end{aligned}$$

proving that $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are orthogonal to each other.

We conclude by the symmetry that $\frac{\partial \mathbf{r}}{\partial u}$, $\frac{\partial \mathbf{r}}{\partial v}$ and $\frac{\partial \mathbf{r}}{\partial w}$ are pairwise orthogonal, and we have proved that (u,v,w) is orthogonal.

4) According to the above,

$$\begin{aligned}h_u^2 &= \left\|\frac{\partial \mathbf{r}}{\partial u}\right\|^2=\frac{1}{(u^2+v^2+w^2)^4}\left\{\left(-u^2+v^2+w^2\right)^2+4u^2v^2+4u^2w^2\right\}\\ &= \frac{1}{(u^2+v^2+w^2)^4}\left\{(u^2)^2-2u^2(v^2+w^2)+(v^2+w^2)+4u^2(v^2+w^2)\right\}\\ &= \frac{1}{(u^2+v^2+w^2)^4}\,(u^2+v^2+w^2)^2,\end{aligned}$$

i.e.

$$h_u^2=\frac{1}{(u^2+v^2+w^2)^2}$$

and hence

$$h_u=\frac{1}{u^2+v^2+w^2}.$$

Then by the symmetry

$$h_v=\frac{1}{u^2+v^2+w^2}\quad\text{and}\quad h_w=\frac{1}{u^2+v^2+w^2}.$$

Example 4.7 *We introduce a set of curvilinear coordinates* (ξ, η, φ) *by*

$$x = \xi\eta\cos\varphi, \quad y = \xi\eta\sin\varphi, \quad z = \frac{1}{2}\left(\xi^2 - \eta^2\right),$$

$$0 \le \xi < +\infty, \quad 0 \le \eta < +\infty, \quad 0 \le \varphi \le 2\pi.$$

1) Show that this defines a rotation coordinate system.

2) Describe the coordinate surfaces, in particular the degenerated ones, $\xi = 0$ *and* $\eta = 0$*. Sketch the meridian half plane.*

3) Show that the coordinate system (ξ, η, φ) *is orthogonal.*

4) Compute the metric coefficients h_ξ*,* h_η*,* h_φ*.*

A Curvilinear coordinates.

D Identify each concept. Apply the theory concerning the relevant formulæ.

I 1) First, φ is eliminated by

$$x^2 + y^2 = (\xi\eta)^2,$$

and we see that z does not depend on φ at all, thus (ξ, η, φ) is a rotation coordinate system.

2) When $\xi = 0$, then $x = y = 0$ and $z = -\frac{1}{2}\,\eta^2$, hence the "coordinate surface" degenerates into the negative Z-axis.

Similarly, when $\eta = 0$ we get $x = y = 0$ and $z = \frac{1}{2}\,\xi^2$, thus the "coordinate surface" degenerates to the positive Z-axis.

Assume e.g. that $\xi \neq 0$ is constant. Then

$$x^2 + y^2 = \xi^2\eta^2, \qquad \text{dvs. } \eta^2 = \frac{1}{\xi^2}\,(x^2 + y^2),$$

hence by insertion

$$z = \frac{1}{2}\,(\xi^2 - \eta^2) = \frac{1}{2}\,\xi^2 - \frac{1}{2\xi^2}\,(x^2 + y^2).$$

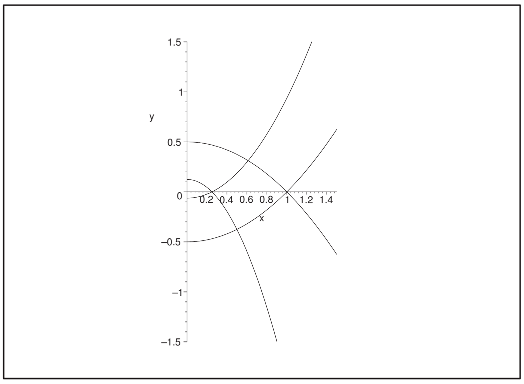
This is the equation of a paraboloid of revolution of vertex $\frac{1}{2}\,\xi^2$ and with the Z-axis as the axis of revolution.

Assume that $\eta \neq 0$ is constant. Then $\xi^2 = \frac{1}{\eta^2}\,(x^2 + y^2)$, thus

$$z = \frac{1}{2}\,(\xi^2 - \eta^2) = \frac{1}{2\eta^2}\,(x^2 + y^2) - \frac{1}{2}\,\eta^2,$$

which is the equation of another paraboloid of revolution of vertex $-\frac{1}{2}\,\eta^2$.

For φ constant we obtain a plane through the Z-axis with e.g. the normal vector $(-\sin\varphi, \cos\varphi, 0)$.



3) We first compute

$$\frac{\partial \mathbf{r}}{\partial \xi} = (\eta\,\cos\varphi, \eta\,\sin\varphi, xi),$$
$$\frac{\partial \mathbf{r}}{\partial \eta} = (\xi\,\cos\varphi, \xi\,\sin\varphi, -\eta),$$
$$\frac{\partial \mathbf{r}}{\partial \varphi} = (-\xi\eta\,\sin\varphi, \xi\eta\cos\varphi, 0) = \xi\eta(-\sin\varphi, \cos\varphi, 0).$$

We derive from these formulæ that

$$\frac{\partial \mathbf{r}}{\partial \xi}\cdot\frac{\partial \mathbf{r}}{\partial \eta} = \xi\eta\,\cos^2\varphi + \xi\eta\,\sin^2\varphi - \xi\eta = 0$$
$$\frac{\partial \mathbf{r}}{\partial \xi}\cdot\frac{\partial \mathbf{r}}{\partial \varphi} = \xi\eta^2(-\sin\varphi\,\cos\varphi + \sin\varphi\,\cos\varphi) + 0 = 0,$$
$$\frac{\partial \mathbf{r}}{\partial \eta}\cdot\frac{\partial \mathbf{r}}{\partial \varphi} = \xi^2\eta(-\sin\varphi\,\cos\varphi + \sin\varphi\,\cos\varphi) + 0 = 0,$$

and the coordinate system (u, v, w) is orthogonal.

4) Finally,

$$h_\xi = \left\|\frac{\partial \mathbf{r}}{\partial \xi}\right\| = \sqrt{\eta^2+\xi^2} = \sqrt{\xi^2+\eta^2},$$
$$h_\eta = \left\|\frac{\partial \mathbf{r}}{\partial \eta}\right\| = \sqrt{\xi^2+\eta^2},$$
$$h_z = \xi\eta.$$

5 Examples from Electromagnetism

Example 5.1 *We consider the following equations for the stationary magnetic field,*

$$\bigtriangledown \times \mathbf{H} = \mathbf{H}, \qquad \mathbf{B} = \bigtriangledown \times \mathbf{A},$$

where $\mathbf{B}$ *is the magnetic flux density,* $\mathbf{H}$ *is the magnetic field intensity,* $\mathbf{A}$ *a magnetic vector potential, and* $\mathbf{J}$ *the electric flow density; the latter is only different from the zero vector in a bounded part of the space. We shall also assume that*

$$r^2\|\mathbf{H}\|,\ r^2\|\mathbf{B}\|,\ r^1\|\mathbf{A}\| \text{ bounded and } \mathbf{B}\cdot\mathbf{H} \geq 0.$$

One can prove that we can attribute to the field the energy

$$W_M = \int \frac{1}{2}\,\mathbf{B}\cdot\mathbf{H}\,d\Omega,$$

where we integrate over the whole space. Show by partial integration that this integral is convergent and that

$$W_M = \int \frac{1}{2}\,\mathbf{J}\cdot\mathbf{A}\,d\Omega,$$

where we shall only integrate over the bounded part of the space, in which $\mathbf{J} \neq \mathbf{0}$.

A Nabla calculus and Electromagnetism.

D Show that $\int \frac{1}{2}\,\mathbf{B}\cdot\mathbf{H}\,d\Omega$ and $\int \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J}\,d\Omega$ both exist. Then reduce $\frac{1}{2}\,\mathbf{B}\cdot\mathbf{H} - \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J}$. (This dirty trick is equivalent to a partial integration).

I Formally, we must assume that all functions and vector functions are of class C^1 in all of $\mathbb{R}^3$ and that they are in particular finite in $\mathbf{0}$. If $\mathbf{J} = \mathbf{0}$ for $r \ge R_0$, then

$$\left|\int \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J}\,d\Omega\right| \le \frac{1}{2}\int_{K(\mathbf{0};R_0)} \|\mathbf{A}\|\cdot\|\mathbf{J}\|\,d\Omega \le \frac{1}{2}\cdot\frac{4\pi R^3}{3}\max_{\|\mathbf{x}\|\le R_0}\|\mathbf{A}(\mathbf{x})\|\cdot\max_{\|\mathbf{x}\le R_0}\|\mathbf{J}(\mathbf{x})\| < +\infty.$$

By using spherical coordinates we get for $R > 1$,

$$\begin{aligned}\left|\int_{K(\mathbf{0};R)\setminus K(\mathbf{0};1)} \frac{1}{2}\,\mathbf{B}\cdot\mathbf{H}\,d\Omega\right| &\le \frac{1}{2}\int_{K(\mathbf{0};R)\setminus K(\mathbf{0};1)} r^2\|\mathbf{B}\|\cdot r^2\|\mathbf{H}\|\cdot\frac{1}{r^4}\,d\Omega\\ &\le C\int_1^R \frac{1}{r^4}\,r^2\,dr = C\left(1-\frac{1}{R}\right) < C,\end{aligned}$$

where C is independent of R, and it follows that the integral is convergent.

It follows from the definitions that

$$\frac{1}{2}\,\mathbf{B}\cdot\mathbf{H} - \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J} = \frac{1}{2}\,(\nabla\times\mathbf{A})\cdot\mathbf{H} - \frac{1}{2}\,\mathbf{A}\cdot(\nabla\times\mathbf{H}) = \frac{1}{2}\;\nabla\cdot(\mathbf{A}\times\mathbf{H}),$$

thus by Gauß's theorem

$$\begin{aligned}(11)&\int_{K(\mathbf{0};R)} \frac{1}{2}\,\mathbf{B}\cdot\mathbf{H}\,d\Omega - \int_{K(\mathbf{0};R)} \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J}\,d\Omega\\ &\quad = \frac{1}{2}\int_{K(\mathbf{0};R)} \nabla\cdot(\mathbf{A}\times\mathbf{H})\,d\Omega = \int_{\partial K(\mathbf{0};R)} \mathbf{n}\cdot(\mathbf{A}\times\mathbf{H})\,dS.\end{aligned}$$

Here we have the estimate

$$\begin{aligned}&\left|\int_{\partial K(\mathbf{0};R)} \mathbf{n}\cdot(\mathbf{A}\times\mathbf{H})\,dS\right| \le \int_{\partial K(\mathbf{0};R)} \frac{R\|\mathbf{A}\|\cdot R^2\|\mathbf{H}\|}{R^3}\,dS\\ &\quad\le C\cdot\frac{1}{R^3}\,\text{areal}(\partial K(\mathbf{0};R)) = C_1\cdot\frac{R^2}{R^3} = C_1\cdot\frac{1}{R}\to 0 \qquad \text{for } R\to+\infty.\end{aligned}$$

Thus by taking the limit $R \to +\infty$ we get from (11) that

$$\int_{\mathbb{R}^3} \frac{1}{2}\,\mathbf{B}\cdot\mathbf{H}\,d\Omega - \int_{\mathbb{R}^3} \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J}\,d\Omega = 0,$$

hence by a rearrangement,

$$W_M = \int_{\mathbb{R}^3} \frac{1}{2}\,\mathbf{B}\cdot\mathbf{H}\,d\Omega = \int_{\mathbb{R}^3} \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J}\,d\Omega = \int_{K(\mathbf{0};R)} \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J}\,d\Omega$$

as required

Example 5.2 *We have for a material which is not in an electric sense an ideal isolator,*

$$\bigtriangledown \cdot \mathbf{D} = \tilde{\varrho}, \qquad \bigtriangledown \cdot \mathbf{J} + \frac{\partial \tilde{\varrho}}{\partial t} = 0, \qquad \mathbf{D} = \alpha\, \mathbf{J},$$

where $\mathbf{D}$ *is the electric flux density,* $\mathbf{J}$ *is the flow density, and* $\tilde{\varrho}$ *is the charge density, while* α *is a scalar field, which describes the electric properties of the material, and* t *is the time. We further assume that we are in a* stationary *case and that we are* given *a current distribution, so* $\mathbf{J}$ *is a known vector field.*
Find an expression of $\tilde{\varrho}$.

A Nabla calculus and Electromagnetism.

D Analyze the equations, when $\mathbf{J}$ and α are given.

I We first derive that

$$\tilde{\varrho} = \bigtriangledown \cdot \mathbf{D} = \bigtriangledown \cdot (\alpha\, \mathbf{J}) = (\bigtriangledown \alpha) \cdot \mathbf{J} + \alpha \bigtriangledown \cdot \mathbf{J}.$$

Since we are in the stationary case, we have

$$\frac{\partial \tilde{\varrho}}{\partial t} = 0,$$

hence

$$\bigtriangledown \cdot \mathbf{J} + \frac{\partial \tilde{\varrho}}{\partial t} = 0$$

implies that $\bigtriangledown \cdot \mathbf{J} = 0$. Finally, by insertion,

$$\tilde{\varrho} = (\bigtriangledown \alpha) \cdot \mathbf{J}.$$

Example 5.3 *Considering potentials it can be proved that the electric field intensity* $\mathbf{E}$ *and the magnetic flux density* B *can be derived from a scalar potential* V *and a vector potential* A *in the following way:*

$$\mathbf{B} = \bigtriangledown \times \mathbf{A}, \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \bigtriangledown V.$$

We also have equations of the same form with another set of potentials $(\tilde{\mathbf{A}}, \tilde{V})$*, provided that*

$$\tilde{V} = V - \frac{\partial g}{\partial t}, \qquad \tilde{\mathbf{A}} + \bigtriangledown g,$$

where g *is a scalar field. We notice that the potentials are not uniquely determined, so it is natural to set up an* extra condition *on the potentials. One often applies the so-called* Lorentz condition

$$\bigtriangledown \cdot \mathbf{A} + \frac{\partial V}{\partial t} = 0.$$

1. *Derive the differential equation which the scalar field* g *must fulfil if one from any given set of potentials* $(\tilde{\mathbf{A}}, \tilde{V})$ *can create a set of potentials* $(\mathbf{A}, V)$*, which also satisfies the Lorentz condition.*

It turns up that one can solve this differential equation. We therefore assume in the following that the Lorentz condition is satisfied, and then consider the vector field

$$\mathbf{Z}(\mathbf{x}, t) = \int_{t_0}^{t} \mathbf{A}(\mathbf{x}, \tau)\, d\tau,$$

where t_0 *is some constant.*

2. *Show that if we put* $V = -\bigtriangledown \cdot \mathbf{Z}$*, then the Lorentz condition is fulfilled.*

3. *Then express the electromagnetic fields* $\mathbf{E}$ *and* $\mathbf{B}$ *by means of the vector field* $\mathbf{Z}$*.*

A Set of potentials satisfying the Lorentz condition.

D Insert into the equations. Note that the operator $\dfrac{\partial}{\partial t}$ commutes with the operators $\bigtriangledown$, $\bigtriangledown\cdot$ and $\bigtriangledown\times$.

I First assume that $(\mathbf{A}, V)$ is given, and let g be a scalar field. If

$$\tilde{V} = V - \frac{\partial g}{\partial t} \quad \text{and} \quad \tilde{\mathbf{A}} + \bigtriangledown g,$$

then

$$\bigtriangledown \times \tilde{\mathbf{A}} = \bigtriangledown \times \mathbf{A} + \bigtriangledown \times \bigtriangledown g = \mathbf{B} + \mathbf{0} = \mathbf{B},$$

and

$$-\frac{\partial \tilde{\mathbf{A}}}{\partial t} - \bigtriangledown \tilde{V} = -\frac{\partial \mathbf{A}}{\partial t} - \bigtriangledown V - \frac{\partial}{\partial t} \bigtriangledown g + \bigtriangledown \left(\frac{\partial g}{\partial t} \right) = \mathbf{E} + \mathbf{0} = \mathbf{E},$$

and we have proved that $(\mathbf{A}, V)$ and $(\tilde{\mathbf{A}}, \tilde{V})$ are both a set of potentials for $\mathbf{B}$ and $\mathbf{E}$.

1) It follows by a rearrangement that

$$V = \tilde{V} + \frac{\partial g}{\partial t} \quad \text{og} \quad \mathbf{A} = \tilde{\mathbf{A}} - \bigtriangledown g,$$

where the set of potentials $(\tilde{\mathbf{A}}, \tilde{V})$ is given. By insertion into the Lorentz condition we get

$$0 = \bigtriangledown \cdot \mathbf{A} + \frac{\partial V}{\partial t} = \bigtriangledown \cdot \tilde{\mathbf{A}} - \bigtriangledown \cdot \bigtriangledown g + \frac{\partial \tilde{V}}{\partial t} + \frac{\partial^2 g}{\partial t^2},$$

and we derive the requested differential equation

$$\bigtriangledown^2 g - \frac{\partial^2 g}{\partial t^2} = \bigtriangledown \cdot \tilde{\mathbf{A}} + \frac{\partial \tilde{V}}{\partial t}$$

where the right hand side is known. This is a classical inhomogeneous wave equation in three space variables and one time variable.

2) Assume that only the vector field $\mathbf{A}$ is given. Put

$$\mathbf{Z}(\mathbf{x}, t) = \int_{t_0}^{t} \mathbf{A}(\mathbf{x}, \tau)\, d\tau, \quad \frac{\partial \mathbf{Z}}{\partial t} = \mathbf{A}, \quad \text{og} \quad V = -\bigtriangledown \cdot \mathbf{Z}.$$

Then

$$\bigtriangledown \cdot \mathbf{A} + \frac{\partial V}{\partial t} = \bigtriangledown \cdot \mathbf{A} - \frac{\partial}{\partial t} (\bigtriangledown \cdot \mathbf{Z}) = \bigtriangledown \cdot \mathbf{A} - \bigtriangledown \cdot \frac{\partial \mathbf{Z}}{\partial t} = \bigtriangledown \cdot \mathbf{A} - \bigtriangledown \cdot \mathbf{A} = 0,$$

and the Lorentz condition is fulfilled.

3) The set of potentials $(\mathbf{A}, V)$ above defines (expressed by $\mathbf{Z}$) the fields $\mathbf{B}$ and $\mathbf{E}$ by the formulæ

$$\mathbf{B} = \bigtriangledown \times \mathbf{A} = \bigtriangledown \times \frac{\partial \mathbf{Z}}{\partial t} = \frac{\partial}{\partial t} (\bigtriangledown \times \mathbf{Z}),$$

and

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \bigtriangledown V = -\frac{\partial^2 \mathbf{Z}}{\partial t^2} + \bigtriangledown(\bigtriangledown \cdot \mathbf{Z}).$$

Example 5.4 *On the figure we are given a normal cut in a* double wire *consisting of two identical, parallel, conductive strip of breadth b and distance a. In the strips are flowing two opposite equally distributed flows. Assuming that the two strips can be considered as infinitely thin and that the permeability μ has the same value everywhere one can show that the inductance of the wire per length $\mathcal{L}$ is given by*

$$\mathcal{L} = \frac{\mu}{\pi b^2} \int_0^b \left\{ \int_0^b \ln \frac{\sqrt{a^2 + (y - \tilde{y})^2}}{|y - \tilde{y}|} \, dy \right\} d\tilde{y}.$$

Consider this as an improper plane integral and find $\mathcal{L}$.

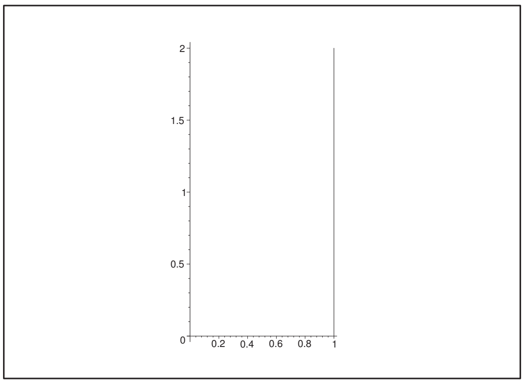


Figure 13: Double wire of distance a and length b.

A Improper plane integral.

D Split the integrand into two parts which each are integrated separately. There is no problem with the first of these integrands. Considering the second one we smooth out the singularity by the first integration.

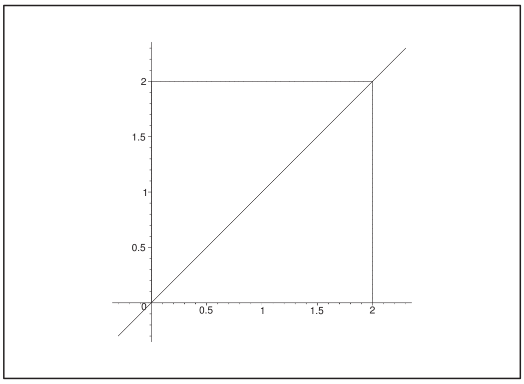


Figure 14: The domain of integration $B = [0, b] \times [0, b]$ for $b = 2$ in the $(y, \tilde{y})$-plane.

I Here, $B = [0, b] \times [0, b]$ in the $(y, \tilde{y})$-plane, and the integrand is not defined for $\tilde{y} = y$. We shall first find an integral of

$$\ln\left(\frac{\sqrt{a^2 + (y-\tilde{y})^2}}{|y-\tilde{y}|}\right) = \frac{1}{2}\ln\left(a^2 + (y-\tilde{y})^2\right) - \frac{1}{2}\ln|y-\tilde{y}|$$

for $\tilde{y}$ fixed and $y \neq \tilde{y}$.

1) When $y \in [0, b]$, then $\frac{1}{2}\ln(a^2 + (y-\tilde{y})^2)$ has no singularity, so we get by a partial integration

$$\begin{aligned}
&\int \frac{1}{2}\ln\left(a^2 + (y-\tilde{y})^2\right) dy \\
&\quad = \frac{1}{2}(y-\tilde{y})\ln\left(a^2 + (y-\tilde{y})^2\right) - \frac{1}{2}\int (y-\tilde{y}) \cdot \frac{2(y-\tilde{y})}{a^2 + (y-\tilde{y})^2}\, dy \\
&\quad = \frac{1}{2}(y-\tilde{y})\ln\left(a^2 + (y-\tilde{y})^2\right) - \int \frac{a^2 + (y-\tilde{y})^2 - a^2}{a^2 + (y-\tilde{y})^2}\, dy \\
&\quad = \frac{1}{2}(y-\tilde{y})\ln\left(a^2 + (y-\tilde{y})^2\right) - y + a\,\mathrm{Arctan}\left(\frac{y-\tilde{y}}{a}\right).
\end{aligned}$$

2) When $\tilde{y} < y \le b$, we get by a partial integration

$$-\frac{1}{2}\int \ln|y-\tilde{y}|\,dy = -\frac{1}{2}\int \ln(y-\tilde{y})\,dy = -\frac{1}{2}\{(y-\tilde{y})\ln(y-\tilde{y})-(y-\tilde{y})\},$$

which due to the order of magnitudes can be extended by 0 for $y=\tilde{y}$.

3) Similarly, we get for $0 \le y < \tilde{y}$ that

$$-\frac{1}{2}\int \ln|y-\tilde{y}|\,dy = -\frac{1}{2}\int \ln(\tilde{y}-y)\,dy = -\frac{1}{2}\{(y-\tilde{y})\ln(\tilde{y}-y)-(y-\tilde{y})\},$$

which is also extended by 0 for $y=\tilde{y}$.

As a conclusion we get from 2) and 3) that

$$-\frac{1}{2}\int \ln|y-\tilde{y}|\,dy = -\frac{1}{2}\,(y-\tilde{y})\ln|y-\tilde{y}| + \frac{1}{2}\,(y-\tilde{y}),$$

which by a continuous extension can be interpreted as 0 for $y=\tilde{y}$.

Thus, for fixed $\tilde{y} \in [0,b]$ we get for the inner integral,

$$\begin{aligned}
&\int_0^b \ln\left(\frac{\sqrt{a^2+(y-\tilde{y})^2}}{|y-\tilde{y}|}\right)dy \\
&= \left[\frac{1}{2}(y-\tilde{y})\ln(a^2+(y-\tilde{y})^2)-y+a\text{Arctan}\left(\frac{y-\tilde{y}}{a}\right)\right]_{y=0}^{b}+\left[-\frac{1}{2}(y-\tilde{y})\ln|y-\tilde{y}|+\frac{1}{2}(y-\tilde{y})\right]_{y=0}^{b} \\
&= -\frac{1}{2}(\tilde{y}-b)\ln((a^2+(\tilde{y}-b)^2)+\frac{1}{2}\tilde{y}\ln(a^2+\tilde{y}^2)-b-a\,\text{Arctan}\left(\frac{\tilde{y}-b}{a}\right)+a\,\text{Arctan}\left(\frac{\tilde{y}}{a}\right) \\
&\qquad +\frac{1}{2}(\tilde{y}-b)\ln(b-\tilde{y})-\frac{1}{2}\tilde{y}\ln\tilde{y}+\frac{1}{2}(b-\tilde{y})+\frac{1}{2}\tilde{y}^2,
\end{aligned}$$

as the singularity has disappeared.

Now put t instead of $\tilde{y}$. Then

$$\begin{aligned}
\mathcal{L} &= \frac{\mu}{\pi b^2}\int_0^b\left\{-\frac{1}{2}(t-b)\ln(a^2+(t-b)^2)\right\}dt+\frac{\mu}{\pi b^2}\int_0^b\frac{1}{2}t\ln(a^2+t^2)\,dt \\
&\quad +\frac{\mu}{\pi b^2}\int_0^b\frac{1}{2}(t-b)\ln|t-b|\,dt-\frac{\mu}{\pi b^2}\int_0^b t\ln t\,dt \\
&\quad -\frac{\mu}{\pi b^2}\,a\int_0^b \text{Arctan}\left(\frac{t-b}{a}\right)dt+\frac{\mu}{\pi b^2}\,a\int_0^b\text{Arctan}\left(\frac{t}{a}\right)dt-\frac{\mu}{\pi b^2}\int_0^b\frac{1}{2}\,b\,dt.
\end{aligned}$$

By some small calculations,

$$\int \tau\ln(k+\tau)\,d\tau = \frac{1}{2}\int\ln(k+\tau^2)\,d\tau^2 = \frac{1}{2}\left\{(k+\tau^2)\ln(k+\tau^2)-(k+\tau^2)\right\},$$

$$\int \tau\ln|\tau|\,d\tau = \frac{1}{2}\tau^2\ln|\tau|-\frac{1}{2}\int\tau\,d\tau = \frac{1}{2}\tau^2\ln|\tau|-\frac{1}{4}\,\tau^2,$$

$$\int \text{Arctan}\left(\frac{\tau}{a}\right) d\tau = \tau \cdot \text{Arctan}\left(\frac{\tau}{a}\right) - \int \frac{\tau}{1+\left(\frac{\tau}{a}\right)^2} \cdot \frac{1}{a}\, d\tau = \tau \ \text{Arctan}\left(\frac{\tau}{a}\right) - \frac{a}{2}\ \ln\left(1+\left(\frac{\tau}{a}\right)^2\right),$$

hence by insertion and convenient choices of τ and k,

$$\begin{aligned}
\mathcal{L} &= \frac{\mu}{\pi b^2}\left\{\left[-\frac{1}{4}(a^2+(t-b)^2)\ln(a^2+(t-b)^2)+\frac{1}{4}(a^2(t-b)^2)\right]_{t=0}^{b}\right. \\
&\quad + \left[\frac{1}{4}(a^2+t^2)\ln(a^2+t^2) - \frac{1}{4}(a^2+t^2)\right]_{t=0}^{b} \\
&\quad + \left[\frac{1}{4}(t-b)^2\ln(b-t) - \frac{1}{8}(t-b)^2\right]_{t=0}^{b} + \left[-\frac{1}{4}t^2\ln t + \frac{1}{8}t^2\right]_{t\to 0}^{b} \\
&\quad + \left[-a(t-b)\text{Arctan}\left(\frac{t-b}{a}\right) + \frac{a^2}{2}\ \ln\left(1+\left(\frac{t-b}{a}\right)^2\right)\right]_{t=0}^{b} \\
&\quad \left. + \left[at\,\text{Arctan}\left(\frac{t}{a}\right) - \frac{a^2}{2}\ln\left\{1+\left(\frac{t}{a}\right)^2\right\}\right]_{t=0}^{b} - \frac{1}{2}b^2\right\} \\
&= \frac{\mu}{\pi b^2}\left\{-\frac{1}{4}a^2\ln(a^2)+\frac{1}{4}a^2 2+\frac{1}{4}(a^2+b^2)\ln(a^2+b^2)\right. \\
&\quad -\frac{1}{4}(a^2+b^2)+\frac{1}{4}(a^2+b^2)\ln(a^2+b^2)-\frac{1}{4}(a^2+b^2)-\frac{1}{4}b^2\ln b+\frac{1}{8}b^2-\frac{1}{4}b^2\ln b \\
&\quad +\frac{1}{8}b^2+ab\,\text{Arctan}\left(\frac{b}{a}\right)-\frac{a^2}{2}\ln\left(\frac{a^2+b^2}{a^2}\right) \\
&\quad \left.+ab\,\text{Arctan}\left(\frac{b}{a}\right)-\frac{a^2}{2}\ln\left(\frac{a^2+b^2}{a^2}\right)-\frac{1}{2}b^2\right\} \\
&= \frac{\mu}{\pi b^2}\left\{-\frac{1}{2}a^2\ln a+\frac{1}{2}(a^2+b^2)\ln(a^2+b^2)-\frac{1}{2}b^2\ln b-a^2\ln(a^2+b^2)\right. \\
&\quad \left.+a^2\ln a+2ab\,\text{Arctan}\left(\frac{b}{a}\right)+\frac{1}{4}a^2-\frac{1}{2}(a^2+b^2)+\frac{1}{4}b^2-\frac{1}{2}b^2\right\} \\
&= \frac{\mu}{\pi b^2}\left\{\frac{1}{2}a^2\ln a-\frac{1}{2}(a^2-b^2)\ln(a^2+b^2)-\frac{1}{2}b^2\ln b+2ab\text{Arctan}\left(\frac{b}{a}\right)-\frac{1}{4}(a^2-b^2)\right\} \\
&= \frac{\mu}{4\pi}\left\{2\left(\frac{a}{b}\right)^2\ln a-2\ln b+2\left[1-\left(\frac{a}{b}\right)^2\right]\ln\left[1+\left(\frac{a}{b}\right)^2\right]+8\,\frac{a}{b}\ \text{Arctan}\left(\frac{b}{a}\right)+1-\left(\frac{a}{b}\right)^2\right\}.
\end{aligned}$$

Example 5.5 *Change the problem of* **Example 5.4** *in the following way: The two parallel strips are placed in the same plane. We assume that the strips may be considered as infinitely thin, that the flows are equally distributed and that the permeability μ is constant. The breadth is denoted by b and the distance by a, cf. the figure. It can be proved that the inductance per length $\mathcal{L}$ of this wire is given by*

$$\mathcal{L} = \frac{\mu}{\pi b^2} \int_{\frac{1}{2}a}^{\frac{1}{2}a+b} \left\{ \int_{\frac{1}{2}a}^{\frac{1}{2}a+b} \ln \frac{x+\tilde{x}}{|x-\tilde{x}|}\, dx \right\} d\tilde{x}.$$

We consider this as an improper plane integral and want to find $\mathcal{L}$. It will be convenient to apply the quotient $\alpha = \dfrac{a}{b}$ and introduce the new variables (ξ, η) by putting

$$x = \frac{1}{2}\,a + b\xi, \qquad \tilde{x} = \frac{1}{2}\,a + b\eta.$$

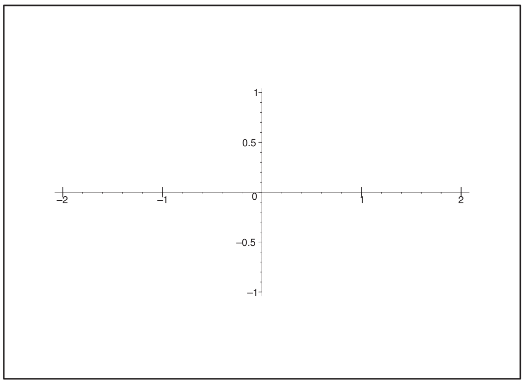


Figure 15: The parallel strips are represented by the intervals $[-2,-1]$ and $[1,2]$, corresponding to $a = 2$ and $b = 1$.

A Improper plane integral and Electromagnetism.

D Sketch the $(x, \tilde{x})$-domain and the (ξ, η)-domain and indicate where the integrand is not defined. Then transform the improper integral into the (ξ, η)-space.

I It follows from $\dfrac{a}{2} \le x = \dfrac{a}{2} + b\xi \le \dfrac{a}{2} + b$ that $0 \le \xi \le 1$, and similarly, $0 \le \eta \le 1$. Furthermore, $\tilde{x} = x$ corresponds to $\xi = \eta$. Finally,

$$\frac{x+\tilde{x}}{|x-\tilde{x}|} = \frac{\dfrac{a}{2} + b\xi + \dfrac{a}{2} + b\eta}{\left|\dfrac{a}{2} + b\xi - \dfrac{a}{2} - b\eta\right|} = \frac{a + b(\xi+\eta)}{b|\xi-\eta|} = \frac{\alpha+\xi+\eta}{|\xi-\eta|} > 1,$$

thus the integrand is positive, and one does not need to be too careful in the computation of the improper plane integral: Either we get $+\infty$, or the right value.

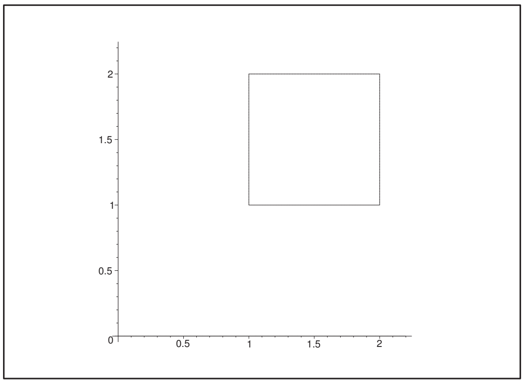


Figure 16: The domain in the $(x, \tilde{x})$-space.

We find

$$
\begin{aligned}
\mathcal{L} &= \frac{\mu}{\pi b^2} \int_{\frac{a}{2}} + \frac{a}{2} + b \left\{ \int_{\frac{a}{2}}^{\frac{a}{2}+b} \ln \frac{x+\tilde{x}}{|x-\tilde{x}|}\, dx \right\} d\tilde{x} = \frac{\mu}{\pi b^2} \int_0^1 \left\{ \int_0^1 \ln \left(\frac{\alpha+\xi+\eta}{|\xi-\eta|} \right) b\, d\xi \right\} b\, d\eta \\
&= \frac{\mu}{\pi} \int_0^1 \left\{ \int_0^1 \ln(\alpha+\xi+\eta)\, d\xi \right\} d\eta - \frac{\mu}{\pi} \int_0^1 \left\{ \int_0^1 \ln|\xi-\eta|\, d\xi \right\} d\eta.
\end{aligned}
$$

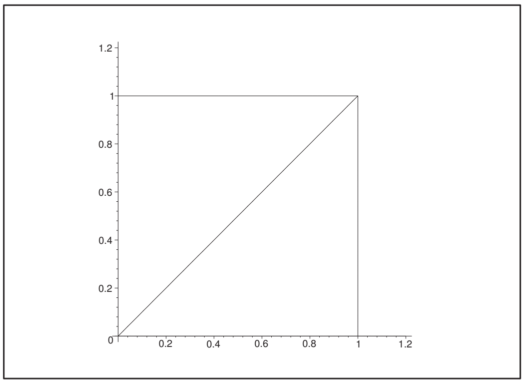


Figure 17: The domain in the (ξ, η)-plane.

Here,

$$\int_0^1 \{\ln(\alpha+\xi+\eta)\,d\xi\}\,d\eta = \int_0^1 [(\alpha+\xi+\eta)\ln(\alpha+\xi+\eta)-(\alpha+\xi+\eta)]_{\xi=0}^1 d\eta$$
$$= \int_0^1 \{(\alpha+1+\eta)\ln(\alpha+1+\eta)-(\alpha+\eta)\ln(\alpha+\eta)-1\}\,d\eta$$
$$= \left[\frac{1}{2}(\alpha+1+\eta)^2\ln(\alpha+1+\eta)-\frac{1}{4}(\alpha+1+\eta)^2-\frac{1}{2}(\alpha+\eta)^2\ln(\alpha+\eta)+\frac{1}{4}(\alpha+\eta)^2\right]_{\eta=0}^1-1$$
$$= \frac{1}{2}(\alpha+2)^2\ln(\alpha+2)-\frac{1}{4}(\alpha+2)^2-\frac{1}{2}(\alpha+1)^2\ln(\alpha+1)+\frac{1}{4}(\alpha+1)^2$$
$$-\frac{1}{2}(\alpha+1)^2\ln(\alpha+1)+\frac{1}{4}(\alpha+1)^2+\frac{1}{2}\alpha^2\ln\alpha-\frac{1}{4}\alpha^2-1$$
$$= \frac{1}{2}(\alpha+2)^2\ln(\alpha+2)-(\alpha+1)^2\ln(\alpha+1)+\frac{1}{2}\alpha^2\ln\alpha-\frac{3}{2}.$$

Then by a symmetry argument,

$$\int_0^1 \left\{\int_0^1 \ln|\xi-\eta|\,d\xi\right\}d\eta = 2\int_0^1\left\{\int_0^\eta \ln(\eta-\xi)\,d\xi\right\}d\eta$$
$$= 2\int_0^1 [(\xi-\eta)\ln(\eta-\xi)-(\xi-\eta)]_{\xi=0}^{\xi\to\eta}d\eta = 2\int_0^1\{\eta\ln\eta+\eta\}\,d\eta$$
$$= 2\left[\frac{\eta^2}{2}\ln\eta-\frac{\eta^2}{4}+\frac{\eta^2}{2}\right]_{\eta\to 0}^1 = 2\cdot\frac{1}{4}=\frac{1}{2}.$$

Thus by insertion,

$$\begin{aligned}\mathcal{L} &= \frac{\mu}{\pi}\left\{\frac{1}{2}(\alpha+2)^2\ln(\alpha+2)-(\alpha+1)^2\ln(\alpha+1)-\frac{3}{2}-\frac{1}{2}\right\}\\ &= \frac{\mu}{2\pi}\left\{(\alpha+2)^2\ln(\alpha+2)-2(\alpha+1)^2\ln(\alpha+1)+\alpha^2\ln\alpha-4\right\}.\end{aligned}$$

Example 5.6 *For an (infinitely) long conductive cylinder with an equally distributed current I where we assume that the permeability μ is constant in space, we get for the magnetic flux density that*

$$\mathbf{B} = \frac{\mu I}{2\pi a^2}\,\mathbf{V},$$

where $\mathbf{V}$ *is the vector field considered in* **Example 2.14**. *We have placed the coordinate system such that the axis of the cylinder is the z-axis, and we describe the cylinder by $\varrho \le a$. The magnetic field intensity* $\mathbf{H}$ *is equal to* $\mathbf{B}/\mu$.

1) Prove that Ampère's law is fulfilled for the considered circles.

2) Show by comparison with **Example 2.14** *that a magnetic vector potential $A\,\mathbf{e}_z$ is given by*

$$A = \begin{cases} \dfrac{\mu I}{4\pi}\left\{1 - \left(\dfrac{\varrho}{a}\right)^2\right\}, & \varrho < a, \\[2mm] \dfrac{\mu I}{2\pi}\,\ln\dfrac{a}{\varrho}, & \varrho \ge a. \end{cases}$$

A Distribution of a current.

D Analyze Ampère's law. The last question is straightforward.

1) Let $\mathbf{H}$ denote the magnetic field intensity and $I(\mathcal{F})$ the electric flow through any surface $\mathcal{F}$. Then by Ampère's law,

$$\oint_{\partial\mathcal{F}} \mathbf{H}\cdot\mathbf{t}\,ds = I(\mathcal{F}).$$

The flow is equally distributed, so the flux density is

$$\mathbf{J} = \begin{cases} \dfrac{\mu I}{\pi a^2}\,\mathbf{e}_z & \text{for } \varrho \le a, \\[2mm] \mathbf{0}, & \text{for } \varrho \ge a, \end{cases}$$

because the area of a cross section of the wire is πa^2.

Now μ and I are constants, so when $\mathcal{F}$ is chosen as a circle ia a plane parallel to the xy-plane and of centrum of the z-axis and of radius ϱ, then

$$\oint_{\mathcal{K}} \mathbf{H}\cdot\mathbf{t}\,ds = I(\mathcal{F}) = \begin{cases} \dfrac{\mu I}{\pi a^2}\,\pi\varrho^2 = \mu I\left(\dfrac{\varrho}{a}\right)^2, & \text{når } \varrho < a, \\[2mm] \dfrac{\mu I}{\pi a^2}\,\pi I, & \text{når } \varrho \ge a. \end{cases}$$

We have for comparison,

$$\oint_{\mathcal{K}} \frac{\mu I}{2\pi a^2}\,\mathbf{V}\cdot\mathbf{t}\,ds = \begin{cases} \dfrac{\mu I}{2\pi a^2}\,2\pi\varrho^2 = \mu I\left(\dfrac{\varrho}{a}\right)^2, & \text{when } \varrho < a, \\ \mu I, & \text{when } \varrho \ge a. \end{cases}$$

We conclude that

(12) $\displaystyle\oint_{\mathcal{K}} \left(\mathbf{H} - \frac{\mu I}{2\pi a^2}\,\mathbf{V}\right) \cdot \mathbf{t}\, ds = 0,$

which is trivially satisfied for

$$\mathbf{H} = \frac{\mu I}{2\pi a^2}\,\mathbf{V}.$$

If we assume that $\mathbf{B} = \mathbf{H}$, we are almost finished. However, see also the following remark.

REMARK. Since $\mathcal{K}$ is chosen among a very special set of curves we can strictly speaking not conclude the uniqueness. However, the existence is obvious. $\Diamond$

2) This is now straightforward.

Example 5.7 *Consider a* double wire*, i.e. two parallel conductive cylinders. The direction of the generator is parallel to the z-axis. We denote the two domains in which the two cylinders intersect the (x, y)-plane by S_1 and S_2. We shall also assume the following:*
The flow density $\mathbf{J}$ of the conductors is parallel to the z-axis, the flows are I and $-I$, and the permeability μ is constant. It can be proved that we get a vector potential $(0, 0, A)$ by adding contributions from the two conductors and that the inductance per length $\mathcal{L}$ is given by

$$\mathcal{L}I^2 = \int_{S_1} JA\, dS + \int_{S_2} JA\, dS.$$

Show by applying the result of **Example 5.6** *and a mean value theorem for harmonic functions that if we consider a double wire consisting of two equal circular cylinders of radius a and distance c $(> 2a)$ between their axis and supporting equally distributed currents, that we have*

$$\mathcal{L} = \frac{\mu}{\pi}\left(\frac{1}{4} + \ln\frac{c}{a}\right).$$

A This is a fairly long example from Electromagnetism with a guideline.

D Sketch a figure. Add the vector potentials from **Example 5.6** in order to find J. Finally, compute $\mathcal{L}$ by showing that some convenient function is harmonic.

I Let S_1 be the disc of centrum $(0, 0)$, and S_2 the disc of centrum $(c, 0)$, both of radius a, where $c > 2a$.

We have according to **Example 5.6**,

$$A_1 = \begin{cases} \dfrac{\mu I}{4\pi}\left\{1 - \dfrac{x^2+y^2}{a^2}\right\} & \text{for } x^2+y^2 < a^2, \\[2ex] \dfrac{\mu I}{4\pi}\ln\left(\dfrac{a^2}{x^2+y^2}\right) & \text{for } x^2+y^2 \geq a^2, \end{cases}$$

and

$$A_2 = \begin{cases} -\dfrac{\mu I}{4\pi}\left\{1 - \dfrac{(x-c)^2+y^2}{a^2}\right\} & \text{for } (x-c)^2+y^2 < a^2, \\[2ex] -\dfrac{\mu I}{4\pi}\ln\left(\dfrac{a^2}{(x-c)^2+y^2}\right) & \text{for } (x-c)^2+y^2 \geq a^2. \end{cases}$$

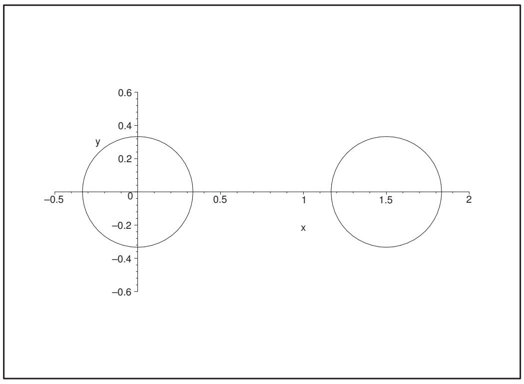


Figure 18: Cross section of the double wire.

Furthermore, $J_1 = \dfrac{I}{\pi a^2}$ and $J_2 = -\dfrac{I}{\pi a^2}$.
Therefore,

$$\begin{aligned}
\mathcal{L} &= \frac{1}{I^2}\int_{S_1} JA\,dS + \frac{1}{I^2}\int_{S_2} JA\,dS \\
&= \frac{1}{I^2}\int_{S_1} J_1(A_1+A_2)dS + \frac{1}{I^2}\int_{S_2} J_2(A_1+A_2)dS \\
&= \frac{1}{I^2}\cdot\frac{I}{\pi a^2}\int_{S_1}\left\{\frac{\mu I}{4\pi}\left(1-\frac{x^2+y^2}{a^2}\right)-\frac{\mu I}{4\pi}\ln\left(\frac{a^2}{(x-c)^2+y^2}\right)\right\}dS \\
&\quad+\frac{1}{I^2}\left(-\frac{I}{\pi a^2}\right)\int_{S_2}\left\{-\frac{\mu I}{4\pi}\left(1-\frac{(x-c)^2+y^2}{a^2}\right)+\frac{\mu I}{4\pi}\ln\left(\frac{a^2}{x^2+y^2}\right)\right\}dS \\
&= \frac{1}{I^2}\cdot\frac{I}{\pi a^2}\cdot\frac{\mu I}{4\pi}\left\{2\int_{S_1}\left(1-\frac{x^2+y^2}{a^2}\right)dS\right. \\
&\quad\left.+\int_{S_1}\left\{\ln\left(\frac{(x-c)^2+y^2}{a^2}\right)+\ln\left(\frac{(x+c=)^2+y^2}{a^2}\right)\right\}dS\right\}.
\end{aligned}$$

The function $f(x,y) = \ln(s^2+y^2)$ is harmonic. In fact,

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2},$$

thus

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2},$$

where the latter follows by either repeating the computation above or by exploiting the symmetry, i.e. by interchanging x and y. Then by adding the results,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

and we have proved that $f(x,y)$ is harmonic. Then

$$\ln\left(\frac{(x\pm c)^2+y^2}{a^2}\right)$$

is also harmonic and we conclude that

$$\begin{aligned}&\int_{S_1}\left\{\ln\left(\frac{(x-c)^2+y^2}{a^2}\right)+\ln\left(\frac{(x+c)^2+y^2}{a^2}\right)\right\}dS\\ &\qquad = \operatorname{area}(S_1)\cdot 2\ln\left(\frac{c^2}{a^2}\right)=4\pi a^2\ln\left(\frac{c}{a}\right).\end{aligned}$$

Furthermore,

$$\int_{S_1}\left(1-\frac{x^2+y^2}{a^2}\right)dS=\pi a^2-\frac{1}{a^2}\int_0^{2\pi}\left\{\int_0^a\varrho^2\cdot\varrho\,d\varrho\right\}d\varphi=\pi a^2-\frac{1}{a^2}\cdot 2\pi\left(\frac{a^4}{4}\right)=\frac{\pi}{2}\,a^2,$$

hence by insertion,

$$\mathcal{L}=\frac{1}{I^2}\cdot\frac{I}{\pi a^2}\cdot\frac{\mu I}{4\pi}\left\{2\cdot\frac{\pi}{2}\,a^2+4\pi a^2\ln\left(\frac{c}{a}\right)\right\}=\frac{\mu}{4\pi}\left\{1+4\ln\left(\frac{c}{a}\right)\right\}=\frac{\mu}{\pi}\left(\frac{1}{4}+\ln\frac{c}{a}\right)$$

as claimed above.

Example 5.8 *A conductive non-magnetic ball K of conductivity γ and radius a is rotating around a diameter of the angular velocity $\mho$ in an homogeneous magnetic field $\mathbf{B}$, which is perpendicular to the vector $\mho$. It can be proved that there is induced a current distribution in the ball with the density*

$$\mathbf{J}=\frac{1}{2}\,\gamma\,\mathbf{x}\times(\mathbf{B}\times\mho),$$

where $\mathbf{x}$ denotes the vector seen from the centrum of the ball. Find the Joule heat effect

$$P=\int_K \frac{J^2}{\gamma}\,d\Omega.$$

A A space integral from Electromagnetism.

D Introduce a convenient coordinate system. Compute $\mathbf{J}$ and then $J^2=\|\mathbf{J}\|^2$. Finally, find P.

I Let K denote the ball of centrum $(0,0,0)$, and assume that it is rotating around the z-axis. Thus for $(x,y,z)\in K\setminus\{(0,0,z)\}$,

$$\mho=\left(-\frac{y}{\sqrt{x^2+y^2}}\,,\,\frac{x}{\sqrt{x^2+y^2}}\,,\,0\right)\omega,$$

where we have put $\omega=\|\mho\|$. Hence, $\mho$ is perpendicular to $\mathbf{e}_z$ everywhere, Therefore, $\mathbf{B}=B\,\mathbf{e}_z$, and we have

$$\mathbf{B}\times\mho=\omega\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ 0 & 0 & B\\ -\dfrac{y}{\sqrt{x^2+y^2}} & \dfrac{x}{\sqrt{x^2+y^2}} & 0\end{vmatrix}=\frac{B\omega}{\sqrt{x^2+y^2}}\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y\\ -y & x\end{vmatrix}=\frac{B\omega}{\sqrt{x^2+y^2}}\,(x,y,0),$$

thus

$$\begin{aligned}\mathbf{J}&=\frac{1}{2}\,\gamma\,\mathbf{x}\times(\mathbf{B}\times\mho)=\frac{1}{2}\,\gamma\cdot\frac{B\omega}{\sqrt{x^2+y^2}}\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ x & y & z\\ x & y & 0\end{vmatrix}\\ &=\frac{1}{2}\,\frac{\gamma B\omega}{\sqrt{x^2+y^2}}\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ 0 & 0 & z\\ x & y & 0\end{vmatrix}=\frac{1}{2}\,\frac{\gamma B\omega}{\sqrt{x^2+y^2}}\,(-z)\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y\\ x & y\end{vmatrix}=-\frac{1}{2}\cdot\frac{\gamma B\omega z}{\sqrt{x^2+y^2}}\,(y,-x,0).\end{aligned}$$

Hence

$$\frac{J^2}{\gamma}=\frac{\|\mathbf{J}\|^2}{\gamma}=\frac{1}{4}\,\gamma\,B^2\omega^2z^2\cdot\frac{x^2+y^2}{x^2+y^2}=\frac{1}{4}\,\gamma\,B^2\omega^2z^2,$$

and thence

$$\begin{aligned}P&=\int_K\frac{J^2}{\gamma}\,d\Omega=\frac{1}{4}\,\gamma B^2\omega^2\int_K z^2\,d\Omega=\frac{1}{4}\,\gamma B^2\omega^2\int_{-a}^{a}z^2(a^2-z^2)\pi\,dz\\ &=\frac{1}{2}\,\gamma B^2\omega^2\pi\int_0^a(a^2z^2-z^4)\,dz=\frac{1}{2}\,\gamma\,B^2\omega^2\pi\left\{\frac{a^5}{3}-\frac{a^5}{5}\right\}=\frac{\pi}{15}\,\gamma\,B^2\omega^2a^5.\end{aligned}$$

6 Miscellaneous

Example 6.1 *The plane domain on the figure S is the union of three rectangles, and it is symmetric with respect to the y-axis.*

1. *Find the barycentre for each of the three rectangles.*

2. *Find the barycentre B for S.*

The dotted line is now used as the y-axis, and the x-axis is put through B.

3. *Compute the* axial moment

$$I_a = \int_S y^2 \, dS.$$

4. *Compute I_a and the area of S for $a = \sqrt{3}$ cm. The moment is given in four decimals.*

A Barycentre and axial moment.

D Find the barycentre and compute the plane integral.

I Assume that S is covered homogeneously. Choose the y-axis as the axis of symmetry, and the lower edge of the figure as the x-axis. Then all three barycentres lie on the y-axis.

1) Put $S = S_1 \cup S_2 \cup S_3$, where S_1 is the upper, S_2 the middle and S_3 the lower rectangle. Clearly, of symmetric reasons,

$$\begin{aligned} y_1 &= a + 5a + \frac{1}{2} \cdot 2a = 7a, && \text{area}(S_1) = 8a^2, \\ y_2 &= a + \frac{1}{2} \cdot 5a = \frac{7}{2}\, a, && \text{area}(S_2) = 5a^2, \\ y_3 &= \frac{1}{2}\, a, && \text{area}(S_3) = 7a^2, \end{aligned}$$

where y_i denotes the ordinate of the corresponding barycentre. We see in particular that

$$\text{area}(S) = (8 + 5 + 7)a^2 = 20a^2.$$

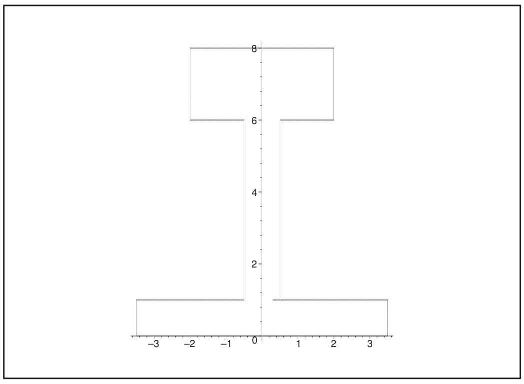


Figure 19: The domain S where the dotted line is replaced by the y-axis and where the lower rectangle has the dimensions 7×1, the rectangle in the middle has the dimensions 1×5 and the upper rectangle has the dimensions 4×2. We have as usual put $a = 1$.

2) Let y denote the ordinate of B. Then

$$y \cdot \text{area}(S) = y_1 \cdot \text{area}(S_1) + y_2 \cdot \text{area}(S_2) + y_3 \cdot \text{area}(S_3),$$

hence

$$y = \frac{a}{20}\left(7 \cdot 8 + \frac{7}{2} \cdot 5 + \frac{1}{2} \cdot 7\right) = \frac{7a}{20}\left(8 + \frac{5}{2} + \frac{1}{2}\right) = \frac{77}{20}\,a.$$

3) Now put the x-axis through B. Then

$$S_1 = [-2a, 2a] \times \left[6a - \frac{77}{20}\,a, 8a - \frac{77}{20}\,a\right] = [-2a, 2a] \times \left[\frac{43}{20}\,a, \frac{83}{20}\,a\right],$$

$$S_2 = \left[-\frac{1}{2}\,a, \frac{1}{2}\,a\right] \times \left[a - \frac{77}{20}\,a, 6a - \frac{77}{20}\,a\right] = \left[-\frac{1}{2}\,a, \frac{1}{2}\,a\right] \times \left[-\frac{57}{20}\,a, \frac{43}{20}\,a\right],$$

$$S_3 = \left[-\frac{7}{2}\,a, \frac{7}{2}\,a\right] \times \left[-\frac{77}{20}\,a, -\frac{57}{20}\,a\right],$$

and the axial moment becomes

$$\begin{aligned}
I_a &= \int_S y^2\,dS = \int_{S_1} y^2\,dS + \int_{S_2} y^2\,dS + \int_{S_3} y^3\,dS \\
&= 4a\left[\frac{y^3}{3}\right]_{\frac{43}{20}a}^{\frac{83}{20}a} + a\left[\frac{y^3}{3}\right]_{-\frac{57}{20}a}^{\frac{43}{20}a} + 7a\left[\frac{y^3}{3}\right]_{-\frac{77}{20}a}^{-\frac{57}{20}a} \\
&= \frac{a^4}{3}\left\{4\left(\frac{83}{20}\right)^3 - 4\left(\frac{43}{20}\right)^3 + \left(\frac{43}{20}\right)^3 + \left(\frac{57}{20}\right)^3 - 7\left(\frac{57}{20}\right)^3 + 7\left(\frac{77}{20}\right)^3\right\} \\
&= \frac{a^4}{3\cdot 20^3}\left\{4\cdot 83^3 - 4\cdot 43^3 + 43^3 + 57^3 - 7\cdot 57^3 + 7\cdot 77^3\right\} \\
&= \frac{a^4}{24000}\left\{4\cdot 83^3 + 7\cdot 77^3 - 3\cdot 43^3 - 6\cdot 57^3\right\} = \frac{4133200}{24000}\,a^4 = \frac{10333}{60}\,a^4.
\end{aligned}$$

4) We get for $a = \sqrt{3}$ cm,

$$\text{area}(S) = 20\cdot 3 = 60 \text{ cm}^2,$$

and

$$I_a = \frac{10333}{60}\cdot 9 = \frac{30999}{20} \approx 1550 \text{ cm}^4.$$

Example 6.2 *Consider for every $a \in \mathbb{R}_+$ the set*

$$L_a = \{(x,y,z) \mid x^2 + y^2 \le az \le a^2\}.$$

1. *Find the volume of L_a.*

2. *Compute the space integral $\int_{L_a} \left(x^2+y^2+z^2\right) d\Omega$.*

Let the vector field $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$\mathbf{V}(x,y,z) = \left(y^2x, z^2y, x^2z\right).$$

3. *Find the flux of $\mathbf{V}$ through the boundary ∂L_a.*

Let $\mathcal{F}_a$ denote that part of ∂L_a, which is given by $x^2+y^2 = az$ and $z \le a$, and let $\mathbf{n}$ denote the outward unit normal vector field of the surface $\mathcal{F}_a$.

4. *Find the flux*

$$\int_{\mathcal{F}_a} \mathbf{n}\cdot\mathbf{rot}\;\mathbf{V}\,dS.$$

Furthermore, let the vector field $\mathbf{W} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$\mathbf{W}(x,y,z) = \left(xz^2, yx^2, zy^2\right),$$

and put $\mathbf{U} = \mathbf{V} + \mathbf{W}$.

5. *Show that the vector field $\mathbf{U}$ is a gradient field and find all its integrals.*

A Volume; mass; flux; gradient field.

D Sketch L_a. Then follow the guidelines. Apply Gauß's theorem, and possibly also Stokes's theorem. Finally, show that $\mathbf{U} \cdot d\mathbf{x}$ is a total differential.

I 1) The set L_a is intersected at the height $z \in [0, a]$ in a disc of area $\pi(x^2 + y^2) = \pi a z$, so we get by the slicing method that

$$\operatorname{vol}(L_a) = \int_0^a a\pi z \, dz = \frac{\pi}{2} a^3.$$

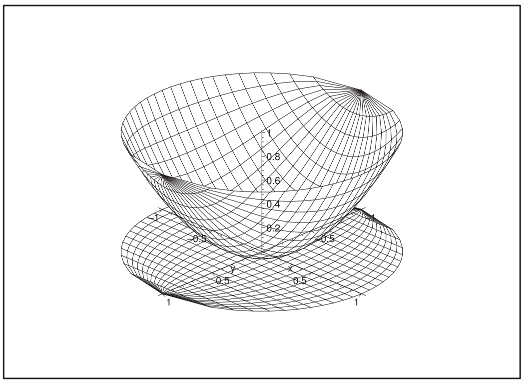


Figure 20: The body L_a and its projection onto the (x, y)-plane for $a = 1$.

2) Put $B_a = \{(x, y) \mid x^2 + y^2 \le a^2\}$. Then we get the integral

$$\begin{aligned}
\int_{L_a} (x^2 + y^2 + z^2) \, d\Omega &= \int_{B_a} \left\{ \int_{\frac{x^2+y^2}{a}}^a (x^2 + y^2 + z^2) \, dz \right\} dx \, dy \\
&= \int_{B_a} \left[(x^2 + y^2) z + \frac{1}{3} z^3 \right]_{z=(x^2+y^2)/a}^a dx \, dy \\
&= \int_{B_a} \left\{ a(x^2 + y^2) + \frac{1}{3} a^3 - \frac{1}{a} (x^2 + y^2)^2 - \frac{1}{3a^3} (x^2 + y^2)^3 \right\} dx \, dy \\
&= 2\pi \int_0^a \left\{ a\varrho^2 + \frac{1}{3} a^3 - \frac{1}{a} \varrho^4 - \frac{1}{3a^3} \varrho^6 \right\} \varrho \, d\varrho \\
&= 2\pi \int_0^a \left\{ \frac{1}{3} a^3 \varrho + a \, \varrho^3 - \frac{1}{a} \varrho^5 - \frac{1}{3a^3} \varrho^7 \right\} d\varrho \\
&= 2\pi \left\{ \frac{1}{6} a^5 + \frac{1}{4} a^5 - \frac{1}{6} a^5 - \frac{1}{24} a^5 \right\} = 2\pi \cdot \frac{5}{24} a^5 = \frac{5\pi}{12} a^5.
\end{aligned}$$

3) Now, $\operatorname{div} \mathbf{V} = y^2 + z^2 + x^2$, so by Gauß's theorem and 2) the flux becomes

$$\operatorname{flux}(\partial L_a) = \int_{L_a} \operatorname{div} \mathbf{V} \, d\Omega = \int_{L_a} (x^2 + y^2 + z^2) \, d\Omega = \frac{5\pi}{12} a^5.$$

4) By Stokes's theorem we get

$$\int_{\mathcal{F}_a} (\mathbf{rot}\ \mathbf{V}) \cdot \mathbf{n}\, dS = \int_{\partial \mathcal{F}_a} \mathbf{V} \cdot \mathbf{t}\, ds,$$

where (cf. the figure)

$$\partial \mathcal{F}_a = \{(x,y,a) \mid x^2+y^2=a^2\} = \{(a\cos t, a\sin t, a) \mid t \in [0,2\pi]\}.$$

Hence along $\partial \mathcal{F}_a$,

$$\mathbf{V}(t) = \left(a^3 \cos t\, \sin^2 t, a^3 \sin t, a^3 \cos^2 t\right), \quad t \in [0,2\pi].$$

When we consult the figure we see that the orientation is pointing in the wrong direction, so in order to obtain an outward normal we must multiply by a factor -1:

$$\begin{aligned}\int_{\mathcal{F}_a} (\mathbf{rot}\ \mathbf{V}) \cdot \mathbf{n}\, dS &= -\int_{\partial \mathcal{F}_a} \mathbf{V} \cdot \mathbf{t}\, ds = -\int_0^{2\pi} \mathbf{V} \cdot (-a\sin t, a\cos t, 0)\, dt \\ &= \int_0^{2\pi} \left\{+a^4 \cos t \cdot \sin^3 t - a^4 \sin t \cdot \cos t + 0\right\} dt = \left[\frac{a^4}{4}\sin^4 t - \frac{a^4}{2}\sin^2 t\right]_0^{2\pi} = 0,\end{aligned}$$

which shows that there has been no need to consider if the orientation was correct.

Alternatively it follows by a straightforward calculation that

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & z^2 y & x^2 z \end{vmatrix} = (-2yz, -2xz, -2xy).$$

By using the parametric description $(x,y,z) = \left(u, v, \dfrac{u^2+v^2}{a}\right)$ of the surface we get

$$\frac{\partial \mathbf{r}}{\partial u} = \left(1, 0, \frac{2u}{a}\right), \qquad \frac{\partial \mathbf{r}}{\partial v} = \left(0, 1, \frac{2v}{a}\right),$$

thus

$$\mathbf{N}_1(u,v) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & \dfrac{2u}{a} \\ 0 & 1 & \dfrac{2v}{a} \end{vmatrix} = \left(-\frac{2u}{a}, -\frac{2v}{a}, 1\right).$$

Since $\mathbf{n}$ is the outward normal field of $\mathcal{F}_a$, it follows by inspection of the figure that the z-coordinate of $\mathbf{n}$ must be negative. We therefore choose

$$\mathbf{N}(u,v) = -\mathbf{N}_1(u,v) = \frac{1}{a}(2u, 2v, -a).$$

Finally, by using a symmetric argument in the computation of the integrals,

$$\begin{aligned}\text{flux}(\mathcal{F}_a) &= \int_{B_a} \textbf{rot}\ \mathbf{V}\cdot\mathbf{n}\,dS = -\frac{1}{a^2}\int_{B_a}\left(2v(u^2+v^2),2u(u^2+v^2),2auv\right)\cdot(-2u,-2v,a)\,du\,dv\\ &= \frac{1}{a^2}\int_{B_a}\left\{4uv(u^2+v^2)+4uv(u^2+v^2)-2a^2uv\right\}du\,dv = 0.\end{aligned}$$

5) First compute the sum

$$\mathbf{U} = \mathbf{V}+\mathbf{W} = \left(y^2x,z^2y,x^2z\right)+\left(xz^2,yx^2,zy^2\right) = \left(x(y^2+z^2),y(x^2+z^2),z(x^2+y^2)\right).$$

This implies

$$\begin{aligned}\mathbf{U}\cdot d\mathbf{x} &= x(y^2+z^2)dx+y(x^2+z^2)dy+z(x^2+y^2)dz\\ &= \frac{1}{2}\left\{(y^2+z^2)d\left(x^2\right)+(x^2+z^2)d\left(y^2\right)+(x^2+y^2)d\left(z^2\right)\right\}\\ &= d\left\{\frac{1}{2}\left(x^2y^2+x^2z^2+y^2z^2\right)\right\},\end{aligned}$$

and we conclude that $\mathbf{U}$ is a gradient field with its integrals given by

$$F(x,y,z) = \frac{1}{2}\left(x^2y^2+y^2z^2+z^2x^2\right)+C,\qquad C\in\mathbb{R}.$$

Example 6.3 *Consider the vector field*

$$\mathbf{V}(x,y)=\left(\frac{xy^2(1+xy)}{1+x^2y^2}\,,\,\frac{x^2y(1+xy)}{1+x^2y^2}\right),\qquad (x,y)\in\mathbb{R}^2.$$

1) Show that $\mathbf{V}$ *is a gradient field and find the integral* $F:\mathbb{R}^2\to\mathbb{R}$*, which is* 0 *at the point* $(0,0)$*.*

2) Write F *as a composite function:*

$$F(x,y)=f(u),\qquad u=g(x,y).$$

3) Find the maximum and the minimum of F *on the set*

$$A=\{(x,y)\mid |x|+|y|\le 2\}$$

by e.g. finding the range $g(A)$ *by a geometric consideration.*

A Gradient field, integral. Maximum and minimum.

D First find an integral. This is here done in three different ways.

I 1) **First variant.** We get by a small manipulation,

$$\begin{aligned}\omega &= \mathbf{V}\cdot d\mathbf{x}=\frac{xy^2(1+xy)}{1+x^2y^2}\,dx+\frac{x^2y(1+xy)}{1+x^2y^2}\,dy\\ &= \frac{xy(1+xy)}{1+x^2y^2}\,(y\,dx+x\,dy)=\frac{1+x^2y^2+xy-1}{1+x^2+y^2}\,d(xy)\\ &= \left(1+\frac{xy}{1+(xy)^2}-\frac{1}{1+(xy)^2}\right)d(xy)\\ &= d\left\{xy+\frac{1}{2}\,\ln\left(1+x^2y^2\right)-\ \text{Arctan}(xy)\right\},\end{aligned}$$

proving that $\mathbf{V}$ is a gradient field with the integrals

$$F_C(x,y)=xy+\frac{1}{2}\,\ln\left(1+x^2y^2\right)-\ \text{Arctan}(xy)+C,\qquad C\in\mathbb{R}.$$

That particular integral which is 0 at $(0,0)$, corresponds to $C=0$, thus

$$F(x,y)=xy+\frac{1}{2}\,\ln\left(1+x^2y^2\right)-\ \text{Arctan}(xy).$$

Second variant. When be integrate along a broken line from $(0,0)$, we get

$$\begin{aligned}F(x,y) &= \int_0^x 0\,dt+\int_0^y\frac{x^2t(1+xt)}{1+x^2t^2}\,dt=\int_0^{xy}\frac{u(1+u)}{1+u^2}\,du\\ &= \int_0^{xy}\frac{1+u^2+u-1}{1+u^2}\,du=\int_0^{xy}\left\{1+\frac{u}{1+u^2}-\frac{1}{1+u^2}\right\}du\\ &= xy+\frac{1}{2}\,\ln\left(1+x^2y^2\right)-\ \text{Arctan}(xy).\end{aligned}$$

C We shall here test the candidate,

$$\begin{aligned} dF &= \left(y+\frac{1}{2}\,\frac{2xy^2}{1+x^2y^2}-\frac{2xy^2}{1+x^2y^2}-\frac{y}{1+x^2y^2}\right)dx \\ &\qquad +\left(x+\frac{1}{2}\,\frac{2x^2y}{1+x^2y^2}-\frac{x}{1+x^2y^2}\right)dy \\ &= \frac{y(1+x^2y^2)+xy^2-y}{1+x^2y^2}\,dx+\frac{x(1+x^2y^2)+x^2y-x}{1+x^2y^2}\,dy \\ &= \frac{xy^2(1+xy)}{1+x^2y^2}\,dx+\frac{x^2y(1+xy)}{1+x^2y^2}\,dy=\mathbf{V}\cdot d\mathbf{x}. \end{aligned}$$

Third variant. We get for y arbitrary,

$$\begin{aligned} F(x,y) &= \int \frac{xy^2(1+xy)}{1+x^2y^2}\,dx=\int\frac{xy(1+xy)}{1+(xy)^2}\,d(xy)=\cdots \\ &= xy+\frac{1}{2}\ln\left(1+x^2y^2\right)-\operatorname{Arctan}(xy), \end{aligned}$$

where the computations follow the same pattern as in the **Second variant**.

C Since $dF=\omega=\mathbf{V}\cdot d\mathbf{x}$, it follows that F is an integral, and as $F(0,0)=0$, the required integral is precisely

$$F(x,y)=xy+\frac{1}{2}\ln\left(1+x^2y^2\right)-\operatorname{Arctan}(xy).$$

2) If we put $u=g(x,y)=xy$, then

$$F(x,y)=f(u)=u+\frac{1}{2}\ln\left(1+u^2\right)-\operatorname{Arctan} u.$$

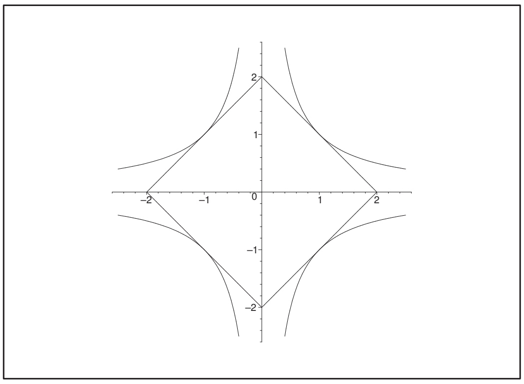


Figure 21: The domain A and the extremal curves $u=xy=\pm 1$.

3) Since $F(x,y)$ is of class C^∞, and A is closed and bounded, it follows from the second main theorem for continuous functions that F has both a maximum and a minimum on A.

It follows from

$$f(u)=u+\frac{1}{2}\ln\left(1+u^2\right)-\operatorname{Arctan} u$$

that (cf. the integrand of the **Second variant**)

$$f'(u) = \frac{u(1+u)}{1+u^2}, \qquad u = xy,$$

which is zero for either $u = 0$ or $u = -1$. Furthermore, f is increasing for $u \in]-\infty, -1[$, decreasing for $u \in]-1, 0[$, and increasing for $u \in]0, +\infty[$.
For $u = xy = 0$ we get

$$F(x,y) = F(0,y) = F(x,0) = 0.$$

For $u = xy = -1$ we get

$$f(-1) = -1 + \frac{1}{2}\ln 2 + \frac{\pi}{4} > 0.$$

In A these correspond to the points $(-1,1)$ and $(1,-1)$.
Let $u = xy = +1$. This corresponds to the points $(1,1)$ and $(-1,-1)$ in A. In this case we get the values

$$f(1) = 1 + \frac{1}{2}\ln 2 - \frac{\pi}{4} > f(-1).$$

We conclude from $u \in [-1,1]$ for $(x,y) \in A$ that the maximum is

$$f(1,1) = f(-1,1) = 1 - \frac{\pi}{4} + \frac{1}{2}\ln 2,$$

and the minimum is

$$f(x,0) = f(0,y) = 0.$$

ALTERNATIVELY we may find the possible stationary points follows by an examination of the boundary.
The possible stationary points satisfy $\mathbf{V}(x,y) = \mathbf{0}$, thus

$$\frac{xy(1+xy)}{1+x^2y^2}\,(y,x) = (0,0).$$

We thus get three possibilities:

$$x = 0, \qquad y = 0, \quad \text{or} \quad xy_{-}1.$$

In the *interior* of A we get

$$\{(x,0) \mid x \in]-2,2[\} \quad \text{and} \quad \{(0,y) \mid y \in]-2,2[\},$$

because the hyperbola $xy = -1$ only intersects A in the boundary points $(1,1)$ and $(-1,1)$ in ∂A.

The boundary is symmetric with respect to $(0,0)$. As $F(-x,-y) = F(x,y)$, it suffices to consider the following boundary curves

$$x + y = 2, \quad x \in [0,2], \quad \text{and} \quad x - y = 2, \quad x \in [0,2].$$

a) If $x+y=2$, i.e. $y=-x+2$, $x\in[0,2]$, we find the restriction

$$\begin{aligned} h_1(x) &= F(x,2-x)\\ &= (2x-x^2)+\frac{1}{2}\,\ln\left(1+\left(2x-x^2\right)^2\right)-\ \text{Arctan}\left(2x-x^2\right) \end{aligned}$$

where

$$\begin{aligned} h_1'(x) &= 2-2x+\frac{1}{2}\cdot\frac{2(2x-x^2)\cdot(2-2x)}{1+(2x-x^2)^2}-\frac{2-2x}{1+(2x-x^2)^2}\\ &= \frac{2(1-x)}{1+(2x-x^2)^2}\cdot\left\{1+(2x-x^2)^2+(2x-x^2)-1\right\}\\ &= \frac{2(1-x)}{1+(2x-x^2)^2}\cdot x(2-x)\{x(2-x)+1\}. \end{aligned}$$

When $x\in\,]0,2[$ this is zero for $x=1$, corresponding to

$$F(1,1)=F(-1,-1)=1+\frac{1}{2}\,\ln 2-\frac{\pi}{4}.$$

b) If $x-y=2$, i.e. $y=x-2$, $x\in[0,2]$, then the restriction is given by

$$h_2(x)=F(x,x-2)=x^2-2x+\frac{1}{2}\,\ln\left(1+\left(x^2-2x\right)^2\right)-\ \text{Arctan}\left(x^2-2x\right)$$

where

$$\begin{aligned}h_2'(x) &= 2x-2+\frac{1}{2}\cdot\frac{2(x^2-2x)\cdot(2x-2)}{1+(x^2-2x)^2}-\frac{2x-2}{1+(x^2-2x)^2}\\ &= \frac{2(x-1)}{1+(x^2-2x)^2}\left\{1+(x^2-2x)^2+(x^2-2x)-1\right\}\\ &= \frac{2(x-1)}{1+(x^2-2x)^2}\cdot x(x-2)\cdot(x-1)^2.\end{aligned}$$

In $x\in\,]0,2[$, this is zero for $x=1$. We get for $x=1$,

$$F(1,-1)=F(-1,1)=-1+\frac{1}{2}\,\ln 2+\frac{\pi}{4}.$$

Finally, we get in the stationary points,

$$F(0,y)=F(x,0)=0.$$

By a numerical comparison of the possible extremum values it follows that the maximum is

$$F(1,1)=F(-1,-1)=1+\frac{1}{2}\,\ln 2-\frac{\pi}{4},$$

and the minimum is

$$F(x,0)=F(0,y)=0.$$

Example 6.4 *Given a C^1-function $U(\mathbf{x})$, $\mathbf{x}\in A$, where $A\subseteq\mathbb{R}^3$, and consider a curve such that $\mathbf{x}$ is a function in time t. The curve is determined by the differential equation*

$$\mathbf{x}''(t)+\bigtriangledown U(\mathbf{x}(t))=0,$$

where $'$ denotes differentiation with respect to t.
Prove by using the chain rule that

$$\frac{1}{2}\,\|\mathbf{x}'\|^2+U=C,$$

where C is a constant. (This differential equation is called a first integral *of the above because the order is reduced by 1).*
In Mechanics, $\mathbf{x}(t)$ can be interpreted as the path of a particle in a field of the potential U; then the two differential equations express Newton's second law and the energy theorem.

A Derivation of the first integral.

D When we analyze the desired result, we see that here occurs $\|\mathbf{x}'\|^2=\|\mathbf{x}'\|\cdot\|\mathbf{x}'\|$, which roughly speaking means that we must have 1 "something like $\mathbf{x}^2$". Hence, the idea must be to take the dot product between the first differential equation and $\mathbf{x}'(t)$ follows by an integration over the parameter interval $I=[t_0,t]$.

I By following the analysis above we get

$$\begin{aligned}
0 &= \int_I \{\mathbf{x}''(t) + \bigtriangledown U(\mathbf{x}(t))\} \cdot \mathbf{x}'(t)\,dt = \int_I \mathbf{x}''(t)\cdot\mathbf{x}'(t)\,dt + \int_I \bigtriangledown U(\mathbf{x}(t))\cdot\mathbf{x}'(t)\,dt \\
&= \int_I \sum_{i=1}^3 x_i''(t)\,c_i'(t)\,dt + \int_I \sum_{i=1}^3 \frac{\partial U}{\partial x_i}\cdot\frac{dx_i}{dt}\,dt = \sum_{i=1}^3 \frac{1}{2}\left[(x_i'(\tau))\right]_{\tau=t_0}^t + \int_{t_0}^t dU(\mathbf{x}(\tau)) \\
&= \frac{1}{2}\,\|\mathbf{x}'(t)\| - c_1 + U(\mathbf{x}(t)) - c_2,
\end{aligned}$$

hence by a rearrangement,

$$\frac{1}{2}\,\|\mathbf{x}'(t)\|^2 + U(\mathbf{x}(t)) = C \qquad \text{(en konstant i } t\text{)}.$$